# Fundamental Solutions and Mapping Properties of Semielliptic Operators

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**Abstract**

An explicit formula is given for a fundamental solution for a class of semielliptic operators. The fundamental solution is used to investigate properties of these operators as mappings between weighted function spaces in $\mathbb{R}^n$. Necessary and sufficient conditions are given for such a mapping to be an isomorphism. Results apply, for example, to elliptic, parabolic, and generalized p-parabolic operators.

**Keywords:** Semielliptic Operator, Fundamental Solution, Isomorphism

## 1 Introduction

Let $\mathcal{L}$ denote a linear *semielliptic* partial differential operator, acting on suitable real or complex $m \times 1$ vector functions $u = u(x)$, $x \in \mathbb{R}^n$, according to

$$\mathcal{L}u = \sum_{\alpha \cdot \gamma = \ell} A_\alpha \partial^\alpha u \quad . \tag{1.1}$$

The coefficients $\{A_\alpha\}$ are constant $m \times m$ matrices with real or complex entries, indexed by multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $\mathbb{R}^n$. The positive integer $\ell$ is the *order* of $\mathcal{L}$,

$$\ell = \max\{|\alpha| : A_\alpha \neq 0\} \quad ,$$

and $\gamma$ is a fixed vector of rational numbers,

$$\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) = \left(\frac{\ell}{\ell_1}, \frac{\ell}{\ell_2}, \ldots, \frac{\ell}{\ell_n}\right) \quad , \tag{1.2}$$

with each $\ell_k$ a positive integer. The *semiellipticity condition* on $\mathcal{L}$ requires that its *symbol*, the matrix of polynomial functions

$$L(x) = \sum_{\alpha \cdot \gamma = \ell} A_\alpha (ix)^\alpha \quad , \tag{1.3}$$

be invertible for all nonzero $x$ in $\mathbb{R}^n$. (Alternative adjectives to *semielliptic*, all used by various authors, are *quasielliptic*, *semi-elliptic*, and *quasi-elliptic*). As explained for example in [11], §2, a consequence of the semiellipticity condition is that $\max_k \ell_k = \ell$, so that $\gamma_k \geq 1$ for each $k$ and $\gamma_k = 1$ for at least one $k$. A further consequence is that, for each $k$, $1 \leq k \leq n$, the term

$$A_{\ell_k e_k} \partial^{\ell_k} / \partial x_k^{\ell_k} \qquad ,$$

corresponding to $\alpha = \ell_k e_k$ where $e_k$ is the $k$th unit coordinate vector in $\mathbb{R}^n$, appears in $\mathcal{L}$ with $A_{\ell_k e_k}$ an invertible matrix. This term is the only unmixed derivative with respect to $x_k$ appearing in $\mathcal{L}$, and $\ell_k$ is the highest order of differentiation with respect to $x_k$ appearing in $\mathcal{L}$.

We consider the subclass of such operators satisfying the additional condition

$$\|\gamma\| := \sum_{k=1}^{n} \gamma_k > \ell \quad . \tag{1.4}$$

We show that for such operators $\mathcal{L}$ a fundamental solution is given explicitly by the iterated integral

$$F(x) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot z} \sigma(z)\, e^{-t\sigma(z)} L(z)^{-1} \; dz \; dt \qquad (x \neq 0) \quad , \tag{1.5}$$

where $\sigma$ is the function

$$\sigma(x) = \sum_{k=1}^{n} {x_k}^{2\ell_k} \quad . \tag{1.6}$$

In particular, $F$ has the properties

- $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
- $\mathcal{L}F(x) = 0$ for $x \neq 0$,
- for all complex $m \times 1$ vector functions $\varphi$ in $C_0^\infty(\mathbb{R}^n)$, $F * \varphi \in C^\infty(\mathbb{R}^n)$ and $\mathcal{L}(F * \varphi) = \varphi$.

It must be pointed out that the order of integration is important in (1.5), as Fubini's Theorem does not apply, and interchanging orders of integration will likely destroy convergence of the integrals.

Of course other methods are known for constructing fundamental solutions for partial differential equations with constant coefficients. Unfortunately these methods do not always produce representations highly useful for investigation of solutions of the corresponding equations.

We use our fundamental solution to investigate properties of the operator $\mathcal{L}$ as a mapping between weighted function spaces. We introduce the vector of integers

$$\underline{\ell} = (\ell_1, \ell_2, \ldots, \ell_n) \quad , \tag{1.7}$$

and a corresponding weight function

$$\rho(x) = \sigma(x)^{1/(2\ell)} = \left( \sum_{k=1}^{n} {x_k}^{2\ell_k} \right)^{1/(2\ell)} \quad . \tag{1.8}$$

We define function spaces $W_s^{r,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, consisting of complex $m \times 1$ vector functions $u$ on $\mathbb{R}^n$ with finite norm

$$\|u\|_{r,p,s;\underline{\ell}} = \sum_{\alpha \cdot \gamma \leq r} \left\| (1+\rho)^{s+\alpha\cdot\gamma} \partial^\alpha u \right\|_{p,\mathbb{R}^n} \quad . \tag{1.9}$$

We demonstrate (Theorem 7.3) that, if $1 < p < \infty$ and $\|\gamma\| > \ell$, then the mapping

$$\mathcal{L} : W_s^{\ell,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}) \longrightarrow W_{s+\ell}^{0,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}) \tag{1.10}$$

is an isomorphism if and only if

$$-\|\gamma\|/p < s < \|\gamma\| - \ell - \|\gamma\|/p \quad . \tag{1.11}$$

Investigations of mappings between weighted Sobolev spaces, analogous to (1.10), began with Cantor [1], who considered the scalar operator $\mathcal{L} = \Delta$, the Laplace operator in $\mathbb{R}^n$. Then $m = 1$, $\ell = 2$, $\gamma = (1, \ldots, 1)$, and the weight function $\rho$ is equivalent to the Euclidean norm. Cantor showed that in this special case (1.10) is an isomorphism provided that $n > 2$, $n/(n-2) < p < \infty$, and $-n/p < s < n - 2 - n/p$. Cantor was building on the work of Walker and Nirenberg [21, 22, 19], who showed that certain elliptic differential operators have finite dimensional null spaces as mappings between (unweighted) Sobolev spaces $W^{\ell,p}(\mathbb{R}^n)$.

McOwen [16] removed Cantor's restriction $p > n/(n-2)$ for the Laplace operator, and further listed various conditions, similar to (1.11), that guarantee a power $\Delta^k$ of the Laplacian is a Fredholm map having certain properties. In particular, for $\Delta^k$ we have $\underline{\ell} = (2k, \ldots, 2k)$, and if $n > 2k$ and $-n/p < s < n - 2k - n/p$, the map

$$\Delta^k : W_s^{2k,p}(\mathbb{R}^n, \mathbb{C}, \underline{\ell}) \longrightarrow W_{s+2k}^{0,p}(\mathbb{R}^n, \mathbb{C}, \underline{\ell})$$

is an isomorphism. Lockhart [14] extended this result to scalar elliptic operators of order $\ell$, with constant coefficients and only highest order terms; for these operators (1.10) is an isomorphism provided that $-n/p < s < n - \ell - n/p$. Lockhart and McOwen [14, 15, 17] consider also elliptic operators with variable coefficients continuous at infinity; for such operators conditions are given under which the mapping (1.10) is Fredholm.

In [10], the author and C. Mawata considered the mapping (1.10) for the case of the heat operator in $\mathbb{R}^n$, $\mathcal{L} = \partial_t - \Delta u$. They gave conditions under which the mapping is Fredholm, and in particular showed that $\mathcal{L}$ is an isomorphism provided that $-(n+2)/p < s < n - (n+2)/p$.

In the last section of the paper are examples demonstrating how the main isomorphism theorem extends known results on elliptic operators, and produces new results for parabolic and generalized r-parabolic operators.

Of special relevance to this work is that of G. V. Demidenko [3, 4] who, working in weighted Sobolev spaces somewhat different from ours, studied mapping properties of the semielliptic opertor (1.1). When translated to the notation of this paper, his result on isomorphic properties asserts that the mapping

$$\mathcal{L} : W^{\ell,p}_{-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \longrightarrow L^p\left(\mathbb{R}^n, \mathbb{C}^m\right)$$

is an isomorphism provided that $1 < p < \|\gamma\| / \ell$. This statement is a special case of our isomorphism result, obtained by taking $s = -\ell$ in (1.10) and (1.11). Demidenko did not use fundamental solutions in his investigations, but rather used integral representations to present what he called "approximate solutions" of equations $\mathcal{L}u = f$, converging in $L^p$ to actual solutions. A modification of Demidenko's representations led to our discovery of formula (1.5) for a fundamental solution.

Demidenko [5, 6, 7] has recently extended his isomorphism results to operators of the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & 0 \end{bmatrix} \quad ,$$

where $\mathcal{L}_1$ is a square matrix semielliptic operator, and $\mathcal{L}_2$ and $\mathcal{L}_3$ are rectangular matrix differential operators having certain properties related to semiellipticity.

# 2 Notation and Preliminaries

In formula (1.1) we use the conventional notation

$$\alpha \cdot \gamma = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_n\gamma_n \qquad , \qquad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad ,$$

with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ representing the *length* of the multi-index $\alpha$. Some authors write (1.1) in the equivalent formulation

$$\mathcal{L}u = \sum_{\alpha/\underline{\ell}=1} A_\alpha \partial^\alpha u \quad ,$$

where $\alpha/\underline{\ell}$ is defined as the sum $\alpha_1/\ell_1 + \cdots + \alpha_n/\ell_n$. However we prefer (1.1), as the vector $\gamma$ proves useful in some of our representations.

For a vector $x \in \mathbb{R}^n$ and for a complex matrix $M = (m_{ij})$, we employ the usual norms

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} \qquad , \qquad |M| = \left(\sum_{i,j} |m_{ij}|^2\right)^{1/2} \quad .$$

We also at times use an alternate norm for vectors, as specified by

$$\|x\| := |x_1| + |x_2| + \cdots + |x_n| \quad ,$$

and as demonstrated already in (1.4). (The length $|\alpha|$ of a multi-index $\alpha$ is not the Euclidean length, but rather the same as $\|\alpha\|$; however we conform to custom and use $|\alpha|$, with the expectation that the correct interpretation will be clear from the context.)

The function $\rho$ of (1.8) serves as an *anisotropic length* of vectors $x$ in $\mathbb{R}^n$. In [11], §2 and §5, one finds verifications of the inequalities

$$\rho(x+y) \le \rho(x) + \rho(y) \qquad , \qquad |x^\alpha| \le \rho(x)^{\alpha\cdot\gamma} \quad . \tag{2.1}$$

For positive real numbers $t$ and for $x \in \mathbb{R}^n$ we denote

$$t^\gamma x := (t^{\gamma_1}x_1, t^{\gamma_2}x_2, \cdots, t^{\gamma_n}x_n) \quad .$$

Straightforward calculations confirm that

$$(t^\gamma x)^\alpha = t^{\alpha\cdot\gamma}x^\alpha \qquad , \qquad \rho\,(t^\gamma x) = t\rho(x) \qquad , \qquad L\,(t^\gamma x) = t^\ell L\,(x) \quad . \tag{2.2}$$

If we fix $x$ in the last two of these inequalities and choose $t = 1/\rho\,(x)$, so that $\rho\,(t^\gamma x) = 1$, we deduce that

$$c_1\,(\mathcal{L})\,\rho(x)^\ell \le |L\,(x)| \le c_2\,(\mathcal{L})\,\rho(x)^\ell \quad , \tag{2.3}$$

$$c_3\,(\mathcal{L})\,\rho(x)^{-\ell} \le \left|L\,(x)^{-1}\right| \le c_4\,(\mathcal{L})\,\rho(x)^{-\ell} \qquad (x \ne 0) \quad , \tag{2.4}$$

where $|\cdots|$ here is the matrix norm, and

$$\begin{aligned} c_1\,(\mathcal{L}) &= \min_{\rho(y)=1} |L(y)| \qquad , \qquad c_2\,(\mathcal{L}) = \max_{\rho(y)=1} |L(y)| \quad , \\ c_3\,(\mathcal{L}) &= \min_{\rho(y)=1} \left|L(y)^{-1}\right| \qquad , \qquad c_4\,(\mathcal{L}) = \max_{\rho(y)=1} \left|L(y)^{-1}\right| \quad . \end{aligned}$$

It is further demonstrated in [11], §2, that, for $x$ in $\mathbb{R}^n$ and multi-indices $\alpha$, there are nonnegative constants $c_5\,(\mathcal{L})$ and $c_6\,(\mathcal{L},\alpha)$ such that

$$|\partial^\alpha L(x)| \le \begin{cases} c_5\,(\mathcal{L})\,\rho\,(x)^{\ell-\alpha\cdot\gamma} & \text{if } \alpha\cdot\gamma \le \ell, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

$$\left|\partial^\alpha L\,(x)^{-1}\right| \le c_6\,(\mathcal{L},\alpha)\,\rho\,(x)^{-\ell-\alpha\cdot\gamma} \qquad (x \ne 0) \quad . \tag{2.6}$$

# 3 A Fundamental Solution

We demonstrate that formula (1.5) does indeed prescribe a fundamental solution for the differential operator $\mathcal{L}$. First we investigage in some detail the inner integral of (1.5). For multi-indices $\beta$ and points $x$ in $\mathbb{R}^n$, and for $t > 0$, we define $m \times m$ matrix valued functions

$$J(x,t)=\int_{\mathbb{R}^n} e^{ix\cdot z}\sigma(z)e^{-t\sigma(z)}L(z)^{-1}\ dz \quad , \tag{3.1}$$

$$J_\beta(x,t)=\int_{\mathbb{R}^n} e^{ix\cdot z}(iz)^\beta\sigma(z)e^{-t\sigma(z)}L(z)^{-1}\ dz \quad . \tag{3.2}$$

Observe that $J_\beta = J$ when $\beta=(0,\ldots,0)$. Also, $J_\beta$ is the formal derivative $\partial_x^\beta J$, arising by differentiation of $J$ under the integral; we will discuss validity of this action, as well as convergence of these integrals and other properties.

We require a lemma concerning convergence of more elementary integrals.

**Lemma 3.1** *Let $s$ be any real constant, and assume $t>0$.*

*(a) The integral*

$$\int_{\rho(x)\geq 1}\rho(x)^s\ dx$$

*is finite if and only if $s<-\|\gamma\|$, in which case*

$$\int_{\rho(x)\geq t}\rho(x)^s\ dx = t^{s+\|\gamma\|}\int_{\rho(x)\geq 1}\rho(x)^s\ dx \quad . \tag{3.3}$$

*(b) The integral*

$$\int_{\rho(x)\leq 1}\rho(x)^s\ dx$$

*is finite if and only if $s>-\|\gamma\|$, in which case*

$$\int_{\rho(x)\leq t}\rho(x)^s\ dx = t^{s+\|\gamma\|}\int_{\rho(x)\leq 1}\rho(x)^s\ dx \quad . \tag{3.4}$$

**Proof.** Given $0\leq r<R\leq\infty$ and $s\in\mathbb{R}$, let $I(r,R,s)$ be the integral

$$I(r,R,s)=\int_{r\leq\rho(x)\leq R}\rho(x)^s\ dx \quad . \tag{3.5}$$

In this integral we make the change of integration parameter $z=t^\gamma x$, where $t$ is a positive constant; then $dz=t^{\|\gamma\|}\ dx$, $\rho(z)=t\rho(x)$, to derive

$$I(tr,tR,s)=t^{s+\|\gamma\|}I(r,R,s) \quad . \tag{3.6}$$

Setting $r=1$, $R=2$, and $t=2^m$ gives

$$I\left(2^m,2^{m+1},s\right)=2^{m(s+\|\gamma\|)}I(1,2,s) \quad .$$

From the geometric summation

$$I(1,\infty,s)=\sum_{m=0}^{\infty}I\left(2^m,2^{m+1},s\right)=I(1,2,s)\sum_{m=0}^{\infty}2^{m(s+\|\gamma\|)}$$

it follows that $I(1,\infty,s)<\infty$ if and only if $s+\|\gamma\|<0$. Then we obtain (3.3) by setting $r=1$, $R=\infty$ in (3.6). Likewise, from

$$I(0,1,s)=\sum_{m=-1}^{-\infty} I\left(2^m,2^{m+1},s\right)=I(1,2,s)\sum_{m=-1}^{-\infty} 2^{m(s+\|\gamma\|)}$$

it follows that $I(0,1,s)<\infty$ if and only if $s+\|\gamma\|>0$, in which case (3.4) is obtained by setting $r=0$, $R=1$ in (3.6). ■

The next lemma is proved in [11], §5.

**Lemma 3.2** *Given real numbers $R$ and $S$ with $0\le R<S$, there exists a real valued function $\psi$ in $C_0^\infty(\mathbb{R}^n)$, with support in the region where $\rho(x)<S$, such that $0\le\psi\le1$, $\psi(x)=1$ if $\rho(x)\le R$, and for any multi-index $\alpha$ and $x$ in $\mathbb{R}^n$,*

$$|\partial^\alpha\psi(x)|\le C(\ell,\alpha)\,(S-R)^{-\alpha\cdot\gamma} \quad .$$

The following lemma, somewhat technical in nature, gathers pertinent information regarding the integrals $J_\beta$.

**Lemma 3.3** *Each integral $J_\beta(x,t)$ converges absolutely for $x\in\mathbb{R}^n$ and $t>0$, with*

$$|J_\beta(x,t)|\le C(\mathcal{L},\beta)\,\frac{t^{-1/2}}{\left[t^{1/(2\ell)}+\rho(x)\right]^{\beta\cdot\gamma+\|\gamma\|}} \quad . \tag{3.7}$$

*Moreover, $J\in C^\infty[\mathbb{R}^n\times(0,\infty)]$, with differentiation of $J$ under the integral of all orders allowed. In particular, for $x\in\mathbb{R}^n$ and $t>0$,*

$$\frac{\partial^\beta}{\partial x^\beta}J(x,t)=J_\beta(x,t) \quad , \tag{3.8}$$

$$\frac{\partial^k}{\partial t^k}\frac{\partial^\beta}{\partial x^\beta}J(x,t)=(-1)^k\int_{\mathbb{R}^n} e^{ix\cdot z}(iz)^\beta\sigma(z)^{k+1}e^{-t\sigma(z)}L(z)^{-1}\;dz \quad , \tag{3.9}$$

*with (3.9) likewise converging absolutely. If also $s>0$, then*

$$J_\beta(x,t)=s^{\beta\cdot\gamma+\|\gamma\|+\ell}J_\beta\left(s^\gamma x,s^{2\ell}t\right) \quad . \tag{3.10}$$

**Proof.** Using (1.8), (2.1), and (2.4), we derive for the integrand of (3.9) the bound

$$\left|e^{ix\cdot z}(iz)^\beta\sigma(z)^{k+1}e^{-t\sigma(z)}L(z)^{-1}\right|\le c_4(\mathcal{L})\,\rho(z)^{\beta\cdot\gamma+2k\ell+\ell}\,e^{-t\sigma(z)} \quad . \tag{3.11}$$

This exponential decay confirms the absolute convergence of (3.9), and (when $k=0$) of $J_\beta(x,t)$.

Next we write

$$J_\beta\left(s^\gamma x,s^{2\ell}t\right)=\int_{\mathbb{R}^n} e^{is^\gamma x\cdot z}(iz)^\beta\sigma(z)e^{-s^{2\ell}t\sigma(z)}L(z)^{-1}\;dz \quad ,$$

and in the integral make the change of variable $y = s^{\gamma} z$, with

$$y^{\beta} = s^{\beta\cdot\gamma} z^{\beta} \quad , \quad \sigma(y) = s^{2\ell}\sigma(z) \quad , \quad L(y) = s^{\ell} L(z) \quad , \quad dy = s^{\|\gamma\|} dz \quad ,$$

to obtain (3.10).

We consider differentiating the integral on the right of (3.9), which we will refer to as $Q(x,t)$, with respect to $x_k$. Recalling that $\left|e^{ir} - 1\right| \le |r|$ for $r \in \mathbb{R}$, we obtain for $h \neq 0$ the estimate

$$\left|\frac{1}{h}\left[Q(x + he_k, t) - Q(x,t)\right]\right| \le \int_{\mathbb{R}^n} |z_k| \left|e^{ixz}(iz)^{\beta}\sigma(z)^{k+1}e^{-t\sigma(z)}L(z)^{-1}\right| \, dz,$$

and note that the latter integral converges in view of (3.11). By the dominated convergence theorem, differentiation of $Q(x,t)$ with respect to $x_k$ under the integral is valid. In particular,

$$\frac{\partial}{\partial x_k} J_{\beta}(x,t) = J_{\beta+e_k}(x,t) \quad ,$$

and an induction argument confirms (3.8). In a similar way, differentiation of $Q(x,t)$ under the integral with respect to $t$ can be justified with use of the inequality $|e^z - 1| \le |z| \, e^{|z|}$; then (3.9) follows by induction.

It remains only to establish the bound (3.7). First we bound the integral

$$J_{\beta}(x,1) = \int_{\mathbb{R}^n} e^{ix\cdot z}(iz)^{\beta}\sigma(z)e^{-\sigma(z)}L(z)^{-1} \, dz \quad . \tag{3.12}$$

By Lemma 3.2, there exists a real valued function $\psi$ in $C_0^{\infty}(\mathbb{R}^n)$, with $0 \le \psi \le 1$, $\psi(z) = 0$ if $\rho(z) \ge 2$, $\psi(z) = 1$ if $\rho(z) \le 1$, and $|\partial^{\alpha}\psi(z)| \le C(\ell,\alpha)$ for any multi-index $\alpha$ in $\mathbb{R}^n$. Given $\varepsilon > 0$, we set

$$\varphi_{\varepsilon}(z) = 1 - \psi\left[\left(\varepsilon^{-1}\right)^{\gamma} z\right] = 1 - \psi\left(\frac{z_1}{\varepsilon^{\gamma_1}}, \frac{z_2}{\varepsilon^{\gamma_2}}, \cdots, \frac{z_n}{\varepsilon^{\gamma_n}}\right) \quad ,$$

so that $\varphi_{\varepsilon} \in C^{\infty}(\mathbb{R}^n)$, $0 \le \varphi_{\varepsilon} \le 1$, and

$$\varphi_{\varepsilon}(z) = \begin{cases} 0 & , \quad \text{if } \rho(z) \le \varepsilon \\ 1 & , \quad \text{if } \rho(z) \ge 2\varepsilon \end{cases} \quad , \quad |\partial^{\alpha}\varphi_{\varepsilon}(z)| \le C(\ell,\alpha)\,\varepsilon^{-\alpha\cdot\gamma} \quad . \tag{3.13}$$

But if $\alpha \neq 0$, $\partial^{\alpha}\varphi_{\varepsilon}(z)$ vanishes except where $\varepsilon \le \rho(z) \le 2\varepsilon$; thus the second inequality of (3.13) implies also

$$|\partial^{\alpha}\varphi_{\varepsilon}(z)| \le C(\ell,\alpha)\,\rho(z)^{-\alpha\cdot\gamma} \quad . \tag{3.14}$$

(For $\alpha = 0$ inequality (3.14) is trivial.)

As (3.12) converges absolutely, for any multi-index $\alpha$ we may write

$$\begin{aligned} (ix)^{\alpha} J_{\beta}(x,1) &= (ix)^{\alpha} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi_{\varepsilon}(z)\, e^{ix\cdot z}(iz)^{\beta}\sigma(z)e^{-\sigma(z)}L(z)^{-1} \, dz \\ &= \lim_{\varepsilon\to 0} \int_{\rho(z)\ge\varepsilon} \left(\partial_z^{\alpha} e^{ix\cdot z}\right)\varphi_{\varepsilon}(z)\,(iz)^{\beta}\sigma(z)e^{-\sigma(z)}L(z)^{-1} \, dz \quad . \end{aligned}$$

We integrate by parts, taking into account the exponential decay of the integrand at infinity (which we discuss in more detail later), as well as the vanishing of $\varphi_\varepsilon(z)$ and all its derivatives on the surface $\rho(z)=\varepsilon$, to obtain

$$\begin{aligned}&(ix)^\alpha J_\beta(x,1)\\ &=(-1)^{|\alpha|}\lim_{\varepsilon\to 0}\int_{\rho(z)\geq\varepsilon} e^{ix\cdot z}\partial^\alpha\left[\varphi_\varepsilon(z)(iz)^\beta\sigma(z)e^{-\sigma(z)}L(z)^{-1}\right]\,dz\ .\end{aligned}\tag{3.15}$$

Now, the derivative

$$\partial^\alpha\left[\varphi_\varepsilon(z)(iz)^\beta\sigma(z)e^{-\sigma(z)}L(z)^{-1}\right]$$

is a finite linear combination of products of the form

$$\partial^\eta\varphi_\varepsilon(z)\ \partial^\mu z^\beta\ \partial^\nu\left[\sigma(z)e^{-\sigma(z)}\right]\ \partial^\tau L(z)^{-1}\quad ,$$

where $\eta$, $\mu$, $\nu$, and $\tau$ are multi-indices in $\mathbb{R}^n$ with $\eta+\mu+\nu+\tau=\alpha$. From (3.14) and (2.6),

$$|\partial^\eta\varphi_\varepsilon(z)|\leq C(\ell,\eta)\rho(z)^{-\eta\cdot\gamma}\quad,\quad\left|\partial^\tau L(z)^{-1}\right|\leq C(\mathcal{L},\tau)\rho(z)^{-\ell-\tau\cdot\gamma}\ .\tag{3.16}$$

In view of the formula

$$\partial^\mu z^\beta=\begin{cases}\frac{\beta!}{(\beta-\mu)!}z^{\beta-\mu} & ,\ \text{if }\mu\leq\beta\ ,\\ 0 & ,\ \text{otherwise}\ ,\end{cases}$$

we may use the second inequality of (2.1) to obtain

$$\left|\partial^\mu z^\beta\right|\leq C(\beta)\rho(z)^{(\beta-\mu)\cdot\gamma}\quad .\tag{3.17}$$

From (1.8) and (2.2), for $t>0$ and any multi-index $\omega$ we infer that

$$\sigma(z)=t^{-2\ell}\sigma(t^\gamma z)\qquad,\qquad\partial^\omega\sigma(z)=t^{\omega\cdot\gamma-2\ell}\partial^\omega\sigma(t^\gamma z)\quad .$$

If $z\neq 0$ we may choose $t=\sigma(z)^{-1/(2\ell)}=\rho(z)^{-1}$, so that $\rho(t^\gamma z)=1$; then we obtain

$$|\partial^\omega\sigma(z)|\leq C(\omega,\underline{\ell})\rho(z)^{2\ell-\omega\cdot\gamma}\quad ,\tag{3.18}$$

where $C(\omega,\underline{\ell})=\sup_{\rho(z)=1}|\partial^\omega\sigma(z)|$.

Any derivative $\partial^\nu\left[\sigma(z)e^{-\sigma(z)}\right]$ is a finite linear combination of terms of the form

$$e^{-\sigma(z)}\left[\prod_{k=1}^K\partial^{\nu^k}\sigma(z)\right]\quad ,$$

where $1\leq K\leq 1+|\nu|\leq 1+|\alpha|$ and $\left\{\nu^k\right\}$ are multi-indices with $\nu^1+\nu^2+\cdots+\nu^K=\nu$. As everything in this linear combination depends upon $\nu$ and $\underline{\ell}$,

we obtain with use of (3.18) the estimate

$$\left|e^{-\sigma(z)}\left[\prod_{k=1}^{K}\partial^{\nu^k}\sigma(z)\right]\right| \leq e^{-\sigma(z)}\prod_{k=1}^{K}C\left(\nu^k,\underline{\ell}\right)\rho(z)^{2\ell-\nu^k\cdot\gamma}$$
$$\leq C(\nu,\underline{\ell})\,\rho(z)^{2K\ell-\nu\cdot\gamma}\,e^{-\sigma(z)} \quad ,$$

$$\left|\partial^{\nu}\left[\sigma(z)\,e^{-\sigma(z)}\right]\right| \leq C(\nu,\underline{\ell})\,e^{-\sigma(z)}\cdot\begin{cases}\rho(z)^{2(1+|\alpha|)\ell-\nu\cdot\gamma} & , \text{ if } \rho(z)\geq 1\ . \\ \rho(z)^{2\ell-\nu\cdot\gamma} & , \text{ if } \rho(z)\leq 1\ .\end{cases}$$

Upon combining this bound with (3.16) and (3.17), while noting that all multi-indices and linear combinations are determined ultimately by $\alpha$, $\beta$, and $\mathcal{L}$, we deduce that

$$\begin{aligned}&\left|\partial^{\alpha}\left[\varphi_{\varepsilon}(z)\,(iz)^{\beta}\,\sigma(z)e^{-\sigma(z)}L(z)^{-1}\right]\right| \\ &\leq C(\mathcal{L},\alpha,\beta)\,e^{-\sigma(z)}\cdot\begin{cases}\rho(z)^{2|\alpha|\ell+\ell+\beta\cdot\gamma-\alpha\cdot\gamma} & , \text{ if } \rho(z)\geq 1\ , \\ \rho(z)^{\ell+\beta\cdot\gamma-\alpha\cdot\gamma} & , \text{ if } 0<\rho(z)\leq 1\ .\end{cases}\end{aligned} \tag{3.19}$$

Note that the displayed exponential decay at infinity justifies our previous integrations by parts. By Lemma 3.1, for integrability of this last expression near $\rho(z)=0$ we require that $\ell+\beta\cdot\gamma-\alpha\cdot\gamma>-\|\gamma\|$. If this condition holds we may let $\varepsilon\to 0$ inside the integral in (3.15), as the bounds (3.19) are independent of $\varepsilon$. We have the pointwise limits $\varphi_{\varepsilon}(z)\to 1$ and $\partial^{\nu}\varphi_{\varepsilon}(z)\to 0$ if $\nu\neq 0$; thus (3.15) results in

$$x^{\alpha}J_{\beta}(x,1)=i^{|\alpha|}\int_{\mathbb{R}^n}e^{ix\cdot z}\partial^{\alpha}\left[(iz)^{\beta}\,\sigma(z)e^{-\sigma(z)}L(z)^{-1}\right]\,dz \quad , \tag{3.20}$$

provided that $\alpha\cdot\gamma<\ell+\beta\cdot\gamma+\|\gamma\|$. Our argument thus far ensures that this integral converges absolutely, and indeed (3.19) and (3.20) imply the bound

$$|x^{\alpha}J_{\beta}(x,1)|\leq C(\mathcal{L},\alpha,\beta) \qquad (\text{ if } \alpha\cdot\gamma<\ell+\beta\cdot\gamma+\|\gamma\|\ ) \quad . \tag{3.21}$$

Now in (3.21) we choose $\alpha=N\ell_k e_k$, where $N$ is a nonnegative integer and $e_k$ is the unit multi-index in the $k$th coordinate direction. The condition $\alpha\cdot\gamma=N\ell_k\gamma_k=N\ell<\ell+\beta\cdot\gamma+\|\gamma\|$ leads to the requirement

$$0\leq N<1+\frac{\beta\cdot\gamma+\|\gamma\|}{\ell} \quad . \tag{3.22}$$

We have $x^{\alpha}={x_k}^{N\ell_k}$, and hence (3.21) gives

$$\left|{x_k}^{N\ell_k}J_{\beta}(x,1)\right|\leq C(\mathcal{L},N,\beta) \quad . \tag{3.23}$$

If $\delta_1$, $\delta_2$, …, $\delta_n$ each have either of the values $+1$ or $-1$, then application of (3.23), and (3.21) with $\alpha=0$, gives

$$\left|\left(1+\delta_1{x_1}^{N\ell_1}+\delta_2{x_2}^{N\ell_2}+\cdots+\delta_n{x_n}^{N\ell_n}\right)J_{\beta}(x,1)\right|\leq C(\mathcal{L},N,\beta) \quad .$$

Now, given any $x$ in $\mathbb{R}^n$, we may choose the values $\{\delta_k\}$ so that this inequality becomes

$$\left(1 + |x_1|^{N\ell_1} + |x_2|^{N\ell_2} + \cdots + |x_n|^{N\ell_n}\right) |J_\beta(x,1)| \le C(\mathcal{L}, N, \beta) \quad .$$

But

$$[1+\rho(x)]^{N\ell} \le C(N,\ell,n)\left(1 + |x_1|^{N\ell_1} + |x_2|^{N\ell_2} + \cdots + |x_n|^{N\ell_n}\right) \quad ,$$

and thus, provided that (3.22) holds,

$$|J_\beta(x,1)| \le \frac{C(\mathcal{L},N,\beta)}{[1+\rho(x)]^{N\ell}} \quad .$$

We may choose an integer $N$ satisfying (3.22) so that $N \ge (\beta\cdot\gamma + \|\gamma\|)/\ell$; then we obtain

$$|J_\beta(x,1)| \le \frac{C(\mathcal{L},\beta)}{[1+\rho(x)]^{\beta\cdot\gamma+\|\gamma\|}} \quad . \tag{3.24}$$

Finally, we set $s = t^{-1/(2\ell)}$ in (3.10) to obtain

$$J_\beta(x,t) = t^{-(\beta\cdot\gamma+\|\gamma\|+\ell)/(2\ell)} J_\beta\left(t^{-\gamma/(2\ell)}x, 1\right) \quad . \tag{3.25}$$

Applying then (3.24), noting that $\rho\left(t^{-\gamma/(2\ell)}x\right) = t^{-1/(2\ell)}\rho(x)$, yields (3.7). ■

Our proposed fundamental solution (1.5) for $\mathcal{L}$ can be written in terms of $J$ as the $m \times m$ matrix valued function

$$F(x) = (2\pi)^{-n}\int_0^\infty J(x,t)\ dt \qquad (x \ne 0) \quad . \tag{3.26}$$

In view of (3.8), the formal derivative $\partial^\beta F$ of (3.26) is

$$F_\beta(x) := (2\pi)^{-n}\int_0^\infty J_\beta(x,t)\ dt \qquad (x \ne 0) \quad . \tag{3.27}$$

We will show that, under the added restriction $\|\gamma\| > \ell$, these integrals converge absolutely if $x \ne 0$. When $x = 0$, (3.27) and (3.25) give

$$F_\beta(0) = (2\pi)^{-n} J_\beta(0,1)\int_0^\infty t^{-(\beta\cdot\gamma+\|\gamma\|+\ell)/(2\ell)}\ dt \quad .$$

As the integral on the right is infinite for any value of $\beta$, formulas (3.26) and (3.27) are undefined at $x = 0$.

**Theorem 3.4** *Suppose $\|\gamma\| > \ell$. Then*

*(a) for $x \ne 0$ each integral (3.27) for $F_\beta(x)$ converges absolutely, and*

$$|F_\beta(x)| \le C(\mathcal{L},\beta)\,\rho(x)^{\ell-\beta\cdot\gamma-\|\gamma\|} \qquad (x \ne 0) \ , \tag{3.28}$$

*(b)* $F \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$, *and* $\partial^{\beta} F(x) = F_{\beta}(x)$ *for all multi-indices* $\beta$ *and all nonzero* $x$ *in* $\mathbb{R}^n$,

*(c) for* $s > 0$ *and* $x \in \mathbb{R}^n$,

$$F_{\beta}(s^{\gamma} x) = s^{\ell - \beta\cdot\gamma - \|\gamma\|} F_{\beta}(x) \quad , \tag{3.29}$$

*(d)* $\mathcal{L}F = 0$ *in the region* $\mathbb{R}^n \setminus \{0\}$.

**Proof.** By (3.7),

$$\int_0^{\infty} |J_{\beta}(x,t)| \ dt \le C(\mathcal{L},\beta) \int_0^{\infty} \frac{t^{-1/2}}{\left[t^{1/(2\ell)} + \rho(x)\right]^{\beta\cdot\gamma + \|\gamma\|}} \ dt \ .$$

For $x \neq 0$ we make the change of integration parameter $t = \rho(x)^{2\ell} s^{2\ell}$, to obtain

$$\int_0^{\infty} |J_{\beta}(x,t)| \ dt \le C(\mathcal{L},\beta)\, \rho(x)^{\ell - \beta\cdot\gamma - \|\gamma\|}\, 2\ell \int_0^{\infty} \frac{s^{\ell-1}}{(s+1)^{\beta\cdot\gamma + \|\gamma\|}} \ ds \quad .$$

The latter integral converges at zero since $\ell \ge 1$, and at infinity since $\beta\cdot\gamma + \|\gamma\| \ge \|\gamma\| > \ell$; thus

$$\int_0^{\infty} |J_{\beta}(x,t)| \ dt \le C(\mathcal{L},\beta)\, \rho(x)^{\ell - \beta\cdot\gamma - \|\gamma\|} \qquad (x \neq 0) \quad . \tag{3.30}$$

Hence (a) follows, with this inequality and (3.27) implying (3.28).

To verify the assertions of (b), for $x \neq 0$ we examine a difference quotient

$$\frac{F_{\beta}(x + s e_k) - F_{\beta}(x)}{s} = (2\pi)^{-n} \int_0^{\infty} \frac{J_{\beta}(x + s e_k, t) - J_{\beta}(x,t)}{s} \ dt \quad . \tag{3.31}$$

If $x \neq 0$ and $|s| < |x|/2$, then the line connecting $x$ to $x + s e_k$ misses the origin, and there is a number $r$ between 0 and $s$ so that

$$\frac{J_{\beta}(x + s e_k, t) - J_{\beta}(x,t)}{s} = \frac{\partial}{\partial x_k} J_{\beta}(x + r e_k, t) = J_{\beta + e_k}(x + r e_k, t) \quad ;$$

then by (3.7),

$$\left| \frac{J_{\beta}(x + s e_k, t) - J_{\beta}(x,t)}{s} \right| \le C(\mathcal{L},\beta) \frac{t^{-1/2}}{\left[t^{1/(2\ell)} + \rho(x + r e_k)\right]^{\beta\cdot\gamma + \gamma_k + \|\gamma\|}} \quad .$$

By the triangle inequality of (2.1), we have $\rho(x + r e_k) \ge \rho(x) - \rho(r e_k) \ge \rho(x)/2$ if $s$ (and thus $r$) is sufficiently small, and we obtain

$$\left| \frac{J_{\beta}(x + s e_k, t) - J_{\beta}(x,t)}{s} \right| \le C(\mathcal{L},\beta) \frac{t^{-1/2}}{\left[t^{1/(2\ell)} + \rho(x)/2\right]^{\beta\cdot\gamma + \gamma_k + \|\gamma\|}} \quad .$$

The condition $\|\gamma\| > \ell$ ensures that the right side of this inequality is an integrable function of $t$ on $(0, \infty)$; as it is also independent of $s$ we may let $s \to 0$ in (3.31) and conclude that

$$\frac{\partial}{\partial x_k} F_\beta(x) = (2\pi)^{-n} \int_0^\infty J_{\beta+e_k}(x,t)\ dt = F_{\beta+e_k}(x) \quad .$$

An induction arguement now confirms (b).

For $s > 0$ and $x \in \mathbb{R}^n$, use of (3.27) and (3.10) gives

$$F_\beta(x) = (2\pi)^{-n} s^{\beta\cdot\gamma+\|\gamma\|+\ell} \int_0^\infty J_\beta\left(s^\gamma x, s^{2\ell} t\right)\ dt \quad .$$

In the last integral we make the change of integration parameter $r = s^{2\ell} t$ to obtain (3.29).

To verify statement (d), we use (1.1), the formula $\partial^\alpha F = F_\alpha$, (3.27), (3.2), and (1.3) to write, for $x \neq 0$,

$$\begin{aligned} \mathcal{L}F(x) &= (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot z} L(z)\,\sigma(z)\, e^{-t\sigma(z)} L(z)^{-1}\ dz\ dt \\ &= I\,(2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot z} \sigma(z)\, e^{-t\sigma(z)}\ dz\ dt \quad , \end{aligned} \tag{3.32}$$

where $I$ is the $m \times m$ identity matrix. It is straightforward to verify that

$$\int_{\mathbb{R}^n} e^{ix\cdot z} \sigma(z)\, e^{-t\sigma(z)}\ dz = -\frac{d}{dt} \int_{\mathbb{R}^n} e^{ix\cdot z} e^{-t\sigma(z)}\ dz \quad ,$$

and so we obtain

$$\mathcal{L}F(x) = I\,(2\pi)^{-n} \left[ \int_{\mathbb{R}^n} e^{ix\cdot z} e^{-t\sigma(z)}\ dz \right]_{t=\infty}^{t=0^+} \quad , \tag{3.33}$$

provided that the evaluations at $t = 0^+$ and $t = \infty$ exist. To address this question we consider a scalar valued function

$$g_k(s,t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{isr} e^{-tr^{2k}}\ dr \qquad (s \in \mathbb{R},\ t > 0) \quad ,$$

where $k$ is a positive integer. According to the discussion in chapter 9, section 2, of the book of Friedman [8], $g_k(s,t)$ is a fundamental solution of the parabolic differential equation

$$\frac{\partial u(s,t)}{\partial t} = (-1)^{k+1} \frac{\partial^{2k} u(s,t)}{\partial s^{2k}} \quad ,$$

and for $s \in \mathbb{R}$ and $t > 0$ satisfies an inequality

$$|g_k(s,t)| \leq C_1(k)\, t^{-1/(2k)} \exp\left[ -C_2(k) \left( \frac{s^{2k}}{t} \right)^{1/(2k-1)} \right] \quad , \tag{3.34}$$

where $C_1(k)$ and $C_2(k)$ are positive constants. (See Theorem 1 in chapter 9 of [8], or §2 of [9] for a more detailed treatment.) In particular, $g_k(s,t)$ vanishes at $t=\infty$, and at $t=0^+$ provided that $s \neq 0$. We now write

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot z} e^{-t\sigma(z)}\ dz = (2\pi)^{-n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{ix_j z_j} e^{-tz_j^{2\ell_j}}\ dz = \prod_{j=1}^{n} g_{\ell_j}(x_j, t)\ ,$$

$$\left| (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot z} e^{-t\sigma(z)}\ dz \right| \leq \prod_{j=1}^{n} \left| g_{\ell_j}(x_j, t) \right| \quad .$$

From the bound (3.34) on $g_k$ we deduce that the product on the right vanishes at $t=\infty$, and at $t=0^+$ provided that $x_j \neq 0$ for some $j$; thus (3.33) gives $\mathcal{L}F(x) = 0$ if $x \neq 0$. ■

We define an integral operator $\mathcal{S}$, prescribed on suitable $m \times 1$ complex vector functions $f$ on $\mathbb{R}^n$ according to

$$\mathcal{S}f(x) = F * f(x) = \int_{\mathbb{R}^n} F(x-y)\ f(y)\ dy \quad . \tag{3.35}$$

**Theorem 3.5** *Assume $\|\gamma\| > \ell$, and let $f$ be an $m \times 1$ complex vector function in the space $C_0^\infty(\mathbb{R}^n)$. Then the integral (3.35) converges absolutely for all $x$ in $\mathbb{R}^n$, and $\mathcal{S}f \in C^\infty(\mathbb{R}^n)$ with*

$$\mathcal{L}(\mathcal{S}f) = f \quad .$$

**Proof.** From (3.35), and (3.28) with $\beta = 0$, we find that

$$|\mathcal{S}f(x)| \leq C(\mathcal{L}) \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|}\ |f(y)|\ dy \quad .$$

Since $f$ has compact support the integral on the right converges at infinity, and by Lemma 3.1 it converges near $y = x$ because $\ell - \|\gamma\| > -\|\gamma\|$. Thus (3.35) converges absolutely.

From (3.35) and (3.26),

$$\mathcal{S}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^\infty J(x-y,t)\ f(y)\ dt\ dy \quad , \tag{3.36}$$

and then from (3.30) with $\beta = 0$,

$$\int_{\mathbb{R}^n} \int_0^\infty |J(x-y,t)\ f(y)|\ dt\ dy \leq C(\mathcal{L}) \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|}\ |f(y)|\ \ dy \quad .$$

As again this integral is finite, we may interchange orders of integration in (3.36) and substitute (3.1) to obtain

$$\mathcal{S}f(x) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot z} \sigma(z) e^{-t\sigma(z)} L(z)^{-1} f(y)\ dz\, dy\, dt. \tag{3.37}$$

Looking at the inner two integrals in (3.37), we use (1.8) and (2.4) to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|e^{i(x-y)\cdot z}\sigma(z)e^{-t\sigma(z)}L\left(z\right)^{-1}f\left(y\right)\right|\;dz\;dy\\
&\leq C\left(\mathcal{L}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(z)^{\ell}\;e^{-t\sigma(z)}\;\;\left|f\left(y\right)\right|\;dz\;dy\\
&\leq C\left(\mathcal{L}\right)\left\|f\right\|_{1,\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(z)^{\ell}\;e^{-t\sigma(z)}\;dz<\infty\quad .
\end{aligned}$$

Thus we may interchange orders of integration in these two integrals to write

$$\mathcal{S}f\left(x\right)=\left(2\pi\right)^{-n/2}\int_0^{\infty}\int_{\mathbb{R}^n}e^{ix\cdot z}\sigma\left(z\right)e^{-t\sigma(z)}L\left(z\right)^{-1}\widehat{f}\left(z\right)\;dz\;dt\quad ,\tag{3.38}$$

where $\widehat{f}$ is the $n$-dimensional Fourier transform of $f$,

$$\widehat{f}\left(z\right)=\left(2\pi\right)^{-n/2}\int_{\mathbb{R}^n}e^{-iy\cdot z}\;f\left(y\right)\;dy\quad .$$

Next we use (2.4) once more to estimate

$$\begin{aligned}
&\int_0^{\infty}\int_{\mathbb{R}^n}\left|e^{ix\cdot z}\sigma\left(z\right)e^{-t\sigma(z)}L\left(z\right)^{-1}\widehat{f}\left(z\right)\right|\;dz\;dt\\
&\leq C\left(\mathcal{L}\right)\int_{\mathbb{R}^n}\rho\left(z\right)^{-\ell}\left|\widehat{f}\left(z\right)\right|\int_0^{\infty}\sigma\left(z\right)e^{-t\sigma(z)}\;dt\;dz\\
&=C\left(\mathcal{L}\right)\int_{\mathbb{R}^n}\rho\left(z\right)^{-\ell}\left|\widehat{f}\left(z\right)\right|\;dz\quad .
\end{aligned}$$

As is well known, if $f\in C_0^{\infty}\left(\mathbb{R}^n\right)$ then $\left|\widehat{f}\left(z\right)\right|$ decreases at infinity faster than any power of $|z|$. Thus the last integral converges at infinity, and by Lemma 3.1 also at zero as we assume $\ell<\|\gamma\|$. Hence we may once more interchange orders of integration in (3.38) to arrive at

$$\begin{aligned}
\mathcal{S}f\left(x\right)&=\left(2\pi\right)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot z}L\left(z\right)^{-1}\widehat{f}\left(z\right)\;\int_0^{\infty}\sigma\left(z\right)e^{-t\sigma(z)}\;dt\;dz\\
&=\;\left(2\pi\right)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot z}L\left(z\right)^{-1}\widehat{f}\left(z\right)dz\quad .
\end{aligned}\tag{3.39}$$

It is an easy manner to check that we may differentiate (3.39) under the integral to obtain, for any multi-index $\alpha$,

$$\partial^{\alpha}\mathcal{S}f\left(x\right)=\left(2\pi\right)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot z}\left(iz\right)^{\alpha}L\left(z\right)^{-1}\widehat{f}\left(z\right)\;dz\quad .\tag{3.40}$$

Indeed, in view of (2.1) and (2.4), absolute convergence of these integrals is confirmed by

$$\int_{\mathbb{R}^n}\left|e^{ix\cdot z}\left(iz\right)^{\alpha}L\left(z\right)^{-1}\widehat{f}\left(z\right)\right|\;dz\leq C\left(\mathcal{L}\right)\int_{\mathbb{R}^n}\rho\left(z\right)^{\alpha\cdot\gamma-\ell}\left|\widehat{f}\left(z\right)\right|\;dz<\infty\quad .$$

Finally, from (1.1) and (3.40) it follows that

$$\begin{aligned}\mathcal{L}\left(\mathcal{S}f\right)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot z} L\left(z\right) L\left(z\right)^{-1} \widehat{f}\left(z\right)\; dz \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot z}\widehat{f}\left(z\right)\; dz = f\left(x\right) \quad ,\end{aligned}$$

with the last equality the Fourier inversion theorem for functions in $C_0^\infty\left(\mathbb{R}^n\right)$. ■

We mention here related integral representations of Demidenko [3, 4], who introduced integral operators $\{P_h\}$ defined by

$$\begin{aligned}&(2\pi)^n P_h f\left(x\right) \\ &= \int_h^{h^{-1}} t^{-\|\gamma\|/\ell}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\left(t^{-\gamma/\ell}z\right)} 2\kappa\sigma\left(z\right)^\kappa e^{-\sigma(z)^\kappa} L\left(z\right)^{-1} f\left(y\right)\; dz\; dy\; dt \;,\end{aligned}$$

where $\kappa$ is a suitable positive integer. Using formulas of Uspenskiĭ [20] regarding certain averagings of functions, Demidenko showed that, as $h \to 0$ and under suitable regularity conditions on $f$, the functions $\{P_h f\}$ converge in a weighted Sobolev norm on $\mathbb{R}^n$ to a solution $u$ of $\mathcal{L}u = f$. A modification of this development leads to the formula for the fundamental solution $F$ and to the integral operator $\mathcal{S}$ (which, when written as a triple integral, closely resembles $P_h$ after some changes in integration parameters).

# 4 Function Spaces

We introduce function spaces useful in working with semielliptic operators.

Given $0 \le r < \infty$, $1 \le p \le \infty$, and a domain $\Omega$ in $\mathbb{R}^n$, we say a complex $m \times 1$ vector function $u$ is in the space $W^{r,p}\left(\Omega, \mathbb{C}^m, \underline{\ell}\right)$ provided that $u$ and its weak derivatives $\partial^\alpha u$, $0 \le \alpha\cdot\gamma \le r$, are in $L^p\left(\Omega\right)$; the norm of $u$ in this space is

$$\|u\|_{r,p;\Omega,\underline{\ell}} = \sum_{\alpha\cdot\gamma\le r}\|\partial^\alpha u\|_{p,\Omega} \quad . \tag{4.1}$$

(We assume always that $\gamma$, $\ell$, and $\underline{\ell}$ are related by (1.2) and (1.7), with $\ell = \max_k\ \ell_k$.) We say that $u \in W^{r,p}_{loc}\left(\Omega, \mathbb{C}^m, \underline{\ell}\right)$ whenever $u \in W^{r,p}\left(\Omega_0, \mathbb{C}^m, \underline{\ell}\right)$ for all bounded open sets $\Omega_0$ with closure in $\Omega$.

If $u$ is defined in all of $\mathbb{R}^n$ and $s$ is a real number, we say $u$ is in the weighted Sobolev space $W^{r,p}_s\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ provided that the weak derivatives $\partial^\alpha u$, $0 \le \alpha\cdot\gamma \le r$, are in $L^p_{loc}\left(\mathbb{R}^n\right)$ and $u$ has finite norm

$$\|u\|_{r,p,s;\underline{\ell}} = \sum_{\alpha\cdot\gamma\le r}\left\|\left(1+\rho\right)^{s+\alpha\cdot\gamma}\partial^\alpha u\right\|_{p,\mathbb{R}^n} \quad . \tag{4.2}$$

Obviously the spaces $W^{r,p}_s\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ are decreasing with respect to $s$; that is

$$s_1 \le s_2 \Longrightarrow W^{r,p}_{s_2}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \subset W^{r,p}_{s_1}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \quad .$$

We are concerned in this paper mainly with the cases $r = \ell$ and $r = 0$. Note that in the space $W_s^{0,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ the norm (4.2) simplifies to

$$\|u\|_{0,p,s;\underline{\ell}} = \|(1+\rho)^s u\|_{p,\mathbb{R}^n} \quad .$$

For $0 < R < S \leq \infty$ we define in $\mathbb{R}^n$ the bounded open sets

$$\Omega(R) = \{x : \rho(x) < R\} \qquad , \qquad \Omega(R,S) = \{x : R < \rho(x) < S\} \quad . \tag{4.3}$$

Following is a density theorem for the spaces $W_s^{r,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$.

**Theorem 4.1** *If $0 \leq r < \infty$, $s \in \mathbb{R}$, and $1 \leq p < \infty$, then $C_0^\infty(\mathbb{R}^n)$ is dense in the space $W_s^{r,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$; that is, given a complex $m \times 1$ vector function $u$ in $W_s^{r,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ and $\varepsilon > 0$, there exists a complex $m \times 1$ vector function $\varphi$ in $C_0^\infty(\mathbb{R}^n)$ such that*

$$\|u - \varphi\|_{r,p,s;\underline{\ell}} = \sum_{\alpha\cdot\gamma \leq r} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\, \partial^\alpha (u - \varphi)\right\|_{p,\mathbb{R}^n} < \varepsilon \quad . \tag{4.4}$$

**Proof.** Let $u$ be as described, and suppose $\varepsilon > 0$. Let $R$ be a real constant, $R \geq 1$. By Lemma 3.2 there exists a real valued function $\psi$ in $C_0^\infty(\mathbb{R}^n)$, with support in the region where $\rho(x) < 2R$, such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ where $\rho(x) \leq R$, and for any multi-index $\alpha$ and $x$ in $\mathbb{R}^n$,

$$|\partial^\alpha \psi(x)| \leq C(\ell, \alpha)\, R^{-\alpha\cdot\gamma} \quad . \tag{4.5}$$

We set $v = \psi u$, so that $v \in W_s^{r,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, $v \equiv u$ where $\rho \leq R$, and $v \equiv 0$ where $\rho \geq 2R$. Then for any multi-index $\alpha$ with $\alpha \cdot \gamma \leq r$,

$$\begin{aligned} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha(v-u)\right\|_{p,\mathbb{R}^n} &= \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha(v-u)\right\|_{p,\Omega(R,\infty)} \\ \leq \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha v\right\|_{p,\Omega(R,2R)} &+ \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha u\right\|_{p,\Omega(R,\infty)} \quad . \end{aligned} \tag{4.6}$$

Use of the product rule for differentiation, along with (4.5), gives

$$\begin{aligned} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha v\right\|_{p,\Omega(R,2R)} &= \left\|(1+\rho)^{s+\alpha\cdot\gamma} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta u\, \partial^{\alpha-\beta}\psi\right\|_{p,\Omega(R,2R)} \\ \leq \sum_{\beta\leq\alpha} &\left\|(1+\rho)^{s+\alpha\cdot\gamma} \binom{\alpha}{\beta} \partial^\beta u\, C(\ell, \alpha-\beta)\, R^{-(\alpha-\beta)\cdot\gamma}\right\|_{p,\Omega(R,2R)} \quad . \end{aligned}$$

Given the requirement $\alpha \cdot \gamma \leq r$ and $\beta \leq \alpha$, there are only a finite number of possible values of $\alpha$ and $\beta$ in these manipulations, depending on $\underline{\ell}$ and $r$. Also, $R \leq 1 + \rho \leq 3R$ in $\Omega(R, 2R)$. It follows that

$$\begin{aligned} &\left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha v\right\|_{p,\Omega(R,2R)} \\ &\leq C(\underline{\ell}, r) \sum_{\beta\leq\alpha} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\beta u\ (1+\rho)^{-(\alpha-\beta)\cdot\gamma}\right\|_{p,\Omega(R,2R)} \\ &\leq C(\underline{\ell}, r) \sum_{\beta\leq\alpha} \left\|(1+\rho)^{s+\beta\cdot\gamma}\, \partial^\beta u\right\|_{p,\Omega(R,\infty)} \quad . \end{aligned}$$

Then from this inequality and (4.6),

$$\left\|(1+\rho)^{s+\alpha\cdot\gamma}\,\partial^{\alpha}(v-u)\right\|_{p,\mathbb{R}^n} \leq C\left(\underline{\ell},r\right)\sum_{\beta\leq\alpha}\left\|(1+\rho)^{s+\beta\cdot\gamma}\,\partial^{\beta}u\right\|_{p,\Omega(R,\infty)} \quad .$$

Since the norm (4.2) is assumed finite, the right side of this last inequality tends to 0 as $R\to\infty$; thus we may choose $R$ large enough that

$$\|v-u\|_{r,p,s;\underline{\ell}} = \sum_{\alpha\cdot\gamma\leq r}\left\|(1+\rho)^{s+\alpha\cdot\gamma}\,\partial^{\alpha}\left(v-u\right)\right\|_{p,\mathbb{R}^n} < \varepsilon/2 \quad . \tag{4.7}$$

Now we use a standard argument involving mollifiers to verify there is a complex $m\times 1$ vector function $\varphi$ in $C_0^{\infty}\left(\mathbb{R}^n\right)$ such that

$$\|\varphi-v\|_{r,p,s;\underline{\ell}} < \varepsilon/2 \quad , \tag{4.8}$$

which when combined with (4.7) yields (4.4). Let $\eta$ be a nonnegative function in $C_0^{\infty}\left(R^n\right)$ vanishing outside the unit ball $|x|\leq 1$, with $\int\eta\;dx=1$. For $t>0$ set $\eta_t(x)=t^{-n}\eta(x/t)$, and let $v_t$ be the convolution $v_t=\eta_t * v$. The support of $v$ lies in some ball of radius $S/2$ centered at 0, and we may assume $S\geq 1$. It follows that $v_t\in C_0^{\infty}\left(R^n\right)$ with support in the ball of radius $S$ about 0 if $t<S/2$. For $\alpha\cdot\gamma\leq r$ we have $\partial^{\alpha}(v_t)=\left(\partial^{\alpha}v\right)_t$ and $\left\|\left(\partial^{\alpha}v\right)_t-\partial^{\alpha}v\right\|_{p,\mathbb{R}^n}\to 0$ as $t\to 0$. For $|x|\leq S$ with $S\geq 1$, crude estimates yield

$$1\leq 1+\rho(x)\leq 1+nS \quad .$$

Therefore, for any $\alpha$ with $\alpha\cdot\gamma\leq r$, as $t\to 0$ we have

$$\left\|(1+\rho)^{s+\alpha\cdot\gamma}\,\partial^{\alpha}\left(v_t-v\right)\right\|_{p,\mathbb{R}^n} \leq C\left(s,r,S,n\right)\left\|\partial^{\alpha}\left(v_t-v\right)\right\|_{p,\mathbb{R}^n} \longrightarrow 0 \quad .$$

Thus, if we let $\varphi=v_t$ we have (4.8) if $t$ is sufficiently small. ∎

Demidenko [2] has also introduced special weighted function spaces for use with semielliptic operators. He defined a space $W_{p,\tau}^{\underline{\ell}}\left(\mathbb{R}^n\right)$, with norm

$$\left\|u,W_{p,\tau}^{\underline{\ell}}\left(\mathbb{R}^n\right)\right\| = \sum_{\alpha\cdot\gamma\leq\ell}\left\|\left(1+\langle x\rangle\right)^{-\tau(1-\alpha\cdot\gamma/\ell)}\,\partial^{\alpha}u\right\|_{p,\mathbb{R}^n} \quad ,$$

where $\langle x\rangle$ is defined by

$$\langle x\rangle^2 = \sum_{k=1}^{n}{x_k}^{2\ell_k} = \rho\left(x\right)^{2\ell} \quad .$$

In terms of $\rho$ this norm is equivalent to

$$\sum_{\alpha\cdot\gamma\leq\ell}\left\|\left(1+\rho\right)^{-\tau(\ell-\alpha\cdot\gamma)}\,\partial^{\alpha}u\right\|_{p,\mathbb{R}^n} \quad .$$

When $\alpha \cdot \gamma = \ell$ the weight reduces to 1, regardless of $\tau$; thus this norm appears fundamentally different from (4.2). However, in the case $\tau = 1$, Demidenko's norm corresponds to our norm $\|u\|_{\ell,p,-\ell;\underline{\ell}}$, and the space $W^{\underline{\ell}}_{p,1}(\mathbb{R}^n)$ is equivalent to $W^{\ell,p}_{-\ell}(\mathbb{R}^n, \mathbb{C}^n, \underline{\ell})$. Demidenko has shown [2] that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{\underline{\ell}}_{p,\tau}(\mathbb{R}^n)$ whenever $0 \le \tau \le 1$.

# 5 Apriori Bound

The next result, taken here as a lemma, is a special case of Theorem 2 of [11].

**Lemma 5.1** *Let $\Omega$ be an open subset in $\mathbb{R}^n$, and let $\Omega_0$ be a bounded open set whose closure lies in $\Omega$. If $1 < p < \infty$ and $\alpha$ is a multi-index with $\alpha \cdot \gamma \le \ell$, then for all complex $m \times 1$ functions $u$ in the space $W^{\ell,p}(\Omega, \mathbb{C}^m, \underline{\ell})$,*

$$\|\partial^\alpha u\|_{p,\Omega_0} \le C(\mathcal{L}, p, \Omega, \Omega_0)\left[\|u\|_{p,\Omega} + \|\mathcal{L}u\|_{p,\Omega}\right] \quad .$$

Following is our fundamental apriori bound regarding the operator $\mathcal{L}$ of (1.1).

**Theorem 5.2** *Let $u$ be a complex $m \times 1$ function in the space $W^{\ell,p}_{loc}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$. If $1 < p < \infty$ and $s \in \mathbb{R}$, then*

$$\begin{aligned} \|u\|_{\ell,p,s;\underline{\ell}} &= \sum_{\alpha\cdot\gamma\le\ell} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\,\partial^\alpha u\right\|_{p,\mathbb{R}^n} \\ &\le C(\mathcal{L}, s, p)\left[\|(1+\rho)^s\,u\|_{p,\mathbb{R}^n} + \left\|(1+\rho)^{s+\ell}\,\mathcal{L}u\right\|_{p,\mathbb{R}^n}\right] \quad . \end{aligned} \tag{5.1}$$

**Proof.** Let $u$, $p$, and $s$ be as described. We assume the right side of (5.1) is finite, as otherwise the inequality is trivial. Let $\alpha$ be a multi-index such that $\alpha \cdot \gamma \le \ell$. We use the notation (4.3).

First, in the region $\Omega(4)$, where $\rho(x) < 4$, Lemma 5.1 implies

$$\|\partial^\alpha u\|_{p,\Omega(2)} \le C(\mathcal{L}, p)\left[\|u\|_{p,\Omega(4)} + \|\mathcal{L}u\|_{p,\Omega(4)}\right] \quad .$$

As $1 \le 1 + \rho(x) \le 5$ in $\Omega(4)$, this inequality implies

$$\begin{aligned} &\int_{\Omega(2)} (1+\rho)^{(s+\alpha\cdot\gamma)p}\,|\partial^\alpha u|^p\ dx \\ &\le C(\mathcal{L}, s, p)\left[\int_{\Omega(4)} (1+\rho)^{sp}\,|u|^p\ dx + \int_{\Omega(4)} (1+\rho)^{(s+\ell)p}\,|\mathcal{L}u|^p\ dx\right] \quad . \end{aligned} \tag{5.2}$$

Next let $t$ be a real constant, $t \ge 1$, and define a function $v$ by

$$v(x) = u(t^\gamma x) \quad .$$

Calculations show that

$$\partial^\alpha v(x) = t^{\alpha\cdot\gamma}\left(\partial^\alpha u\right)\left(t^\gamma x\right) \qquad , \qquad \mathcal{L}v(x) = t^\ell\left(\mathcal{L}u\right)\left(t^\gamma x\right) \quad .$$

Again by Lemma 5.1,

$$\int_{\Omega(2,4)} \left|\partial^\alpha v(x)\right|^p \, dx \leq C\left(\mathcal{L},p\right)\left[\int_{\Omega(1,8)} |v(x)|^p \, dx + \int_{\Omega(1,8)} |\mathcal{L}v(x)|^p \, dx\right] \quad ,$$

or in terms of $u$,

$$\begin{aligned}&\int_{\Omega(2,4)} \left|t^{\alpha\cdot\gamma}\left(\partial^\alpha u\right)\left(t^\gamma x\right)\right|^p \, dx \\ &\leq C\left(\mathcal{L},p\right)\left[\int_{\Omega(1,8)} \left|u\left(t^\gamma x\right)\right|^p \, dx + \int_{\Omega(1,8)} \left|t^\ell\left(\mathcal{L}u\right)\left(t^\gamma x\right)\right|^p \, dx\right] \quad .\end{aligned}$$

In these last integrals we make the change of integration parameter $y = t^\gamma x$, with $\rho(y) = t\rho(x)$, $dy = t^{\|\gamma\|}\, dx$, and obtain

$$\begin{aligned}&t^{(\alpha\cdot\gamma)p}\int_{\Omega(2t,4t)} \left|\left(\partial^\alpha u\right)(y)\right|^p \, dy \\ &\leq C\left(\mathcal{L},p\right)\left[\int_{\Omega(t,8t)} |u\left(y\right)|^p \, dy + t^{\ell p}\int_{\Omega(t,8t)} |\mathcal{L}u(y)|^p \, dy\right] \quad .\end{aligned}$$

But in $\Omega(t,8t)$ with $t \geq 1$, we have $t \leq 1+\rho(y) \leq 9t$, and so we may multiply this inequality by $t^{sp}$ to obtain

$$\begin{aligned}&\int_{\Omega(2t,4t)} (1+\rho)^{(s+\alpha\cdot\gamma)p}\left|\partial^\alpha u\right|^p \, dy \\ &\leq C\left(\mathcal{L},s,p\right)\left[\int_{\Omega(t,8t)} (1+\rho)^{sp}|u|^p \, dy + \int_{\Omega(t,8t)} (1+\rho)^{(s+\ell)p}|\mathcal{L}u|^p \, dy\right] \quad .\end{aligned}$$

Now in this inequality we take $t = 2^m$ for $m = 0,1,2,\ldots$, and add all the resulting inequalities to (5.2) to arrive at

$$\begin{aligned}&\int_{\mathbb{R}^n} (1+\rho)^{(s+\alpha\cdot\gamma)p}\left|\partial^\alpha u\right|^p \, dy \\ &\leq C\left(\mathcal{L},s,p\right)\left[\int_{\mathbb{R}^n} (1+\rho)^{sp}|u|^p \, dy + \int_{\mathbb{R}^n} (1+\rho)^{(s+\ell)p}|\mathcal{L}u|^p \, dy\right] \quad ,\end{aligned}$$

which leads to

$$\left\|(1+\rho)^{s+\alpha\cdot\gamma}\,\partial^\alpha u\right\|_{p,\mathbb{R}^n} \leq C\left(\mathcal{L},s,p\right)\left[\left\|(1+\rho)^s\,u\right\|_{p,\mathbb{R}^n} + \left\|(1+\rho)^{s+\ell}\,\mathcal{L}u\right\|_{p,\mathbb{R}^n}\right] \quad .$$

Finally, we sum over all $\alpha$ such that $\alpha\cdot\gamma \leq \ell$ to obtain (5.1). ■

## 6 The Operator $\mathcal{S}$

We investigate properties of the operator $\mathcal{S}$ as a mapping between certain function spaces. We require another technical lemma.

**Lemma 6.1** *For $x$ in $\mathbb{R}^n$, and for real numbers $\xi$ and $\eta$, let*

$$K(x,\xi,\eta) = \int_{\mathbb{R}^n} \rho(x-y)^{\xi} [1+\rho(y)]^{\eta} \; dy \quad .$$

*If*

$$\xi + \|\gamma\| > 0 \quad , \quad \eta + \|\gamma\| > 0 \quad , \quad \xi + \eta + \|\gamma\| < 0 \quad , \tag{6.1}$$

*then*

$$K(x,\xi,\eta) \le C(\xi,\eta,\underline{\ell}) [1+\rho(x)]^{\xi+\eta+\|\gamma\|} \quad . \tag{6.2}$$

**Proof.** Note that conditions (6.1) imply that $\xi, \eta < 0$.

Fixing $x$ in $\mathbb{R}^n$, we partition $\mathbb{R}^n$ into three disjoint regions,

$$\begin{aligned} R_1 &= \left\{ y : \rho(x-y) \le \frac{1+\rho(x)}{2} \right\} \quad , \\ R_2 &= \left\{ y : \frac{1+\rho(x)}{2} < \rho(x-y) < 2[1+\rho(x)] \right\} \quad , \\ R_3 &= \{ y : 2[1+\rho(x)] \le \rho(x-y) \} \quad , \end{aligned}$$

and write

$$K(x,\xi,\eta) = K_1(x,\xi,\eta) + K_2(x,\xi,\eta) + K_3(x,\xi,\eta) \quad ,$$

where

$$K_i(x,\xi,\eta) = \int_{R_i} \rho(x-y)^{\xi} [1+\rho(y)]^{\eta} \; dy \quad , \quad i = 1,2,3 \quad .$$

As $\rho$ satisfies the triangle inequality, in the region $R_1$ we have

$$\begin{gathered} 1+\rho(x) \le 1+\rho(y)+\rho(x-y) \le 1+\rho(y) + \frac{1+\rho(x)}{2} \quad , \\ 1+\rho(x) \le 2[1+\rho(y)] \quad . \end{gathered}$$

We use the fact that $\eta < 0$, along with Lemma 3.1(b) and $\xi > -\|\gamma\|$, to derive

$$\begin{aligned} K_1(x,\xi,\eta) &\le \left[\frac{1+\rho(x)}{2}\right]^{\eta} \int_{\rho(x-y)\le[1+\rho(x)]/2} \rho(x-y)^{\xi} \; dy \\ &\le 2^{-\eta}[1+\rho(x)]^{\eta} \int_{\rho(z)\le 1+\rho(x)} \rho(z)^{\xi} \; dz \\ &= 2^{-\eta}[1+\rho(x)]^{\eta}[1+\rho(x)]^{\xi+\|\gamma\|} \int_{\rho(z)\le 1} \rho(z)^{\xi} \; dz \\ &= C(\xi,\eta,\underline{\ell})[1+\rho(x)]^{\xi+\eta+\|\gamma\|} \quad . \end{aligned}$$

In the region $R_2$,

$$\rho(y) \leq \rho(x-y) + \rho(x) \leq 2\left[1+\rho(x)\right] + \rho(x) \leq 3\left[1+\rho(x)\right] \quad .$$

We use $\xi < 0$ and $-\|\gamma\| < \eta < 0$, along with Lemma 3.1(b), to derive

$$\begin{aligned} K_2(x,\xi,\eta) &\leq \int_{[1+\rho(x)]/2<\rho(x-y)<2[1+\rho(x)]} \left[\frac{1+\rho(x)}{2}\right]^{\xi} \left[1+\rho(y)\right]^{\eta} \; dy \\ &\leq 2^{-\xi}\left[1+\rho(x)\right]^{\xi} \int_{\rho(y)\leq 3[1+\rho(x)]} \rho(y)^{\eta} \; dy \\ &= C(\xi,\eta,\underline{\ell})\left[1+\rho(x)\right]^{\xi+\eta+\|\gamma\|} \quad . \end{aligned}$$

In the region $R_3$,

$$\begin{aligned} \rho(x-y) &\leq \rho(x)+\rho(y) \leq \frac{\rho(x-y)}{2} + \rho(y) \quad , \\ \rho(x-y) &\leq 2\rho(y) \leq 2\left[1+\rho(y)\right] \quad . \end{aligned}$$

We use the fact that $\eta < 0$, along with Lemma 3.1(a) and $\xi + \eta < -\|\gamma\|$, to derive

$$\begin{aligned} K_3(x,\xi,\eta) \leq & \int_{2[1+\rho(x)]\leq\rho(x-y)} \rho(x-y)^{\xi} \left[\frac{\rho(x-y)}{2}\right]^{\eta} \; dy \\ &\leq 2^{-\eta} \int_{1+\rho(x)\leq\rho(z)} \rho(z)^{\xi+\eta} \; dz \leq C(\xi,\eta,\underline{\ell})\left[1+\rho(x)\right]^{\xi+\eta+\|\gamma\|} \quad . \end{aligned}$$

Combining finally our estimates for $K_1$, $K_2$, and $K_3$ gives (6.2). ■

**Lemma 6.2** *Suppose $\|\gamma\| > \ell$, $1 \leq p \leq \infty$, let $s$ be a real number in the range*

$$\ell - \|\gamma\|/p < s < \|\gamma\| - \|\gamma\|/p \quad , \tag{6.3}$$

*and let $f$ be a complex $m \times 1$ vector function such that $\|(1+\rho)^s f\|_{p,\mathbb{R}^n} < \infty$. Then the integral*

$$\mathcal{S}f(x) = F * f(x) = \int_{\mathbb{R}^n} F(x-y) \; f(y) \; dy \quad . \tag{6.4}$$

*converges absolutely for almost all $x$ in $\mathbb{R}^n$, and*

$$\left\|(1+\rho)^{s-\ell}\,\mathcal{S}f\right\|_{p,\mathbb{R}^n} \leq C(\mathcal{L},p,s)\,\|(1+\rho)^s f\|_{p,\mathbb{R}^n} \quad . \tag{6.5}$$

*If moreover $p > \|\gamma\|/\ell$, then in fact (6.4) converges absolutely for all $x$ in $\mathbb{R}^n$, and*

$$|\mathcal{S}f(x)| \leq C(\mathcal{L},p,s)\left[1+\rho(x)\right]^{\ell-s-\|\gamma\|/p} \|(1+\rho)^s f\|_{p,\mathbb{R}^n} \quad . \tag{6.6}$$

**Proof.** From (6.4), and (3.28) with $\beta = 0$,

$$\begin{aligned} |\mathcal{S}f(x)| &\le \int_{\mathbb{R}^n} |F(x-y)| \ |f(y)| \ dy \\ &\le C(\mathcal{L}) \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|} |f(y)| \ dy, \end{aligned} \tag{6.7}$$

First consider the case $1 < p < \infty$. Let $q$ be defined by the usual relation $1/p + 1/q = 1$. In general, two finite and nonempty open intervals $(a, b)$ and $(c, d)$ intersect if and only if $a < d$ and $c < b$. Condition (6.3) implies

$$\frac{\ell}{q} < s + \frac{\|\gamma\| - \ell}{p} \qquad , \qquad s < \frac{\|\gamma\|}{q} \quad ;$$

thus there is a real number $r$ such that

$$\frac{\ell}{q} < r < \frac{\|\gamma\|}{q} \qquad , \qquad s < r < s + \frac{\|\gamma\| - \ell}{p} \quad . \tag{6.8}$$

By (6.7) and Hölder's inequality,

$$\begin{aligned} |\mathcal{S}f(x)| \le C(\mathcal{L}) &\left( \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|} [1 + \rho(y)]^{-rq} \ dy \right)^{1/q} \\ &\cdot \left( \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|} [1 + \rho(y)]^{rp} |f(y)|^p \ dy \right)^{1/p} \quad . \end{aligned}$$

As (6.8) and $\ell \ge 1$ imply that $\xi = \ell - \|\gamma\|$ and $\eta = -rq$ satisfy the hypotheses of Lemma 6.1, we may apply that result and conclude that

$$\begin{aligned} |\mathcal{S}f(x)|^p \le\, & C(\mathcal{L}, p, s) [1 + \rho(x)]^{-rp + \ell p/q} \\ &\cdot \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|} [1 + \rho(y)]^{rp} |f(y)|^p \ dy \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} \left[ \left\| (1+\rho)^{s-\ell} \mathcal{S}f \right\|_{p, \mathbb{R}^n} \right]^p &= \int_{\mathbb{R}^n} [1 + \rho(x)]^{(s-\ell)p} |\mathcal{S}f(x)|^p \ dx \\ &\le C(\mathcal{L}, p, s) \int_{\mathbb{R}^n} [1 + \rho(y)]^{rp} |f(y)|^p \\ &\quad \cdot \int_{\mathbb{R}^n} \rho(x-y)^{\ell - \|\gamma\|} [1 + \rho(x)]^{sp - rp - \ell} \ dx \ dy \ . \end{aligned}$$

Again, (6.8) implies that $\xi = \ell - \|\gamma\|$ and $\eta = sp - rp - \ell$ satisfy the hypotheses of Lemma 6.1; applying that lemma (with the roles of $x$ and $y$ reversed), we obtain

$$\left[ \left\| (1+\rho)^{s-\ell} \mathcal{S}f \right\|_{p, \mathbb{R}^n} \right]^p \le C(\mathcal{L}, p, s) \int_{\mathbb{R}^n} [1 + \rho(y)]^{sp} |f(y)|^p \ dy \quad ,$$

and thereby (6.5). Note that we have verified that the integral on the right of (6.7) defines a function of $x$ in the space $L^p_{loc}(\mathbb{R}^n)$. In particular, for almost all $x$ this integral must be finite, and consequently the integral defining $\mathcal{S}f(x)$ absolutely convergent.

The case $p = 1$, when (6.3) reduces to $\ell - \|\gamma\| < s < 0$, is simpler. We multiply (6.7) by $[1+\rho(x)]^{s-\ell}$, integrate over $\mathbb{R}^n$ with respect to $x$, and apply Lemma 6.1 as above to obtain (6.5) with $p = 1$.

For the case $p = \infty$, when (6.3) reduces to $\ell < s < \|\gamma\|$, we apply first (6.7) and then Lemma 6.1 to derive

$$\begin{aligned}|\mathcal{S}f(x)| &\leq C(\mathcal{L}) \|(1+\rho)^s f\|_\infty \int_{\mathbb{R}^n} \rho(x-y)^{\ell-\|\gamma\|} |1+\rho(y)|^{-s} \ dy \\ &\leq C(\mathcal{L}, s) \|(1+\rho)^s f\|_\infty [1+\rho(x)]^{\ell-s} \quad .\end{aligned}$$

This inequality implies (6.5), as well as (6.6), for the case $p = \infty$.

Next assume $\|\gamma\|/\ell < p < \infty$. Application of Hölder's inequality to (6.7) gives

$$|\mathcal{S}f(x)| \leq C(\mathcal{L}) \left( \int_{\mathbb{R}^n} \rho(x-y)^{(\ell-\|\gamma\|)q} [1+\rho(y)]^{-sq} \ dy \right)^{1/q} \|(1+\rho)^s f\|_{p,\mathbb{R}^n} \ .$$

Conditions (6.3), and $p > \|\gamma\|/\ell$, ensure that Lemma 6.1 applies with $\xi = (\ell - \|\gamma\|)q$ and $\eta = -sq$, resulting in (6.6). ■

**Theorem 6.3** *Suppose $\|\gamma\| > \ell$, $1 < p < \infty$, and let $s$ be a real number in the range*

$$-\|\gamma\|/p < s < \|\gamma\| - \ell - \|\gamma\|/p \quad . \tag{6.9}$$

*Let $f$ be a complex $m \times 1$ vector function in the space $W^{0,p}_{s+\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$; i.e, such that $\left\|(1+\rho)^{s+\ell} f\right\|_{p,\mathbb{R}^n} < \infty$. Then the integral*

$$\mathcal{S}f(x) = \int_{\mathbb{R}^n} F(x-y) \ f(y) \ dy \quad .$$

*converges absolutely for almost all $x$ in $\mathbb{R}^n$, and $\mathcal{S}f \in W^{\ell,p}_s(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ with $\mathcal{L}(\mathcal{S}f) = f$ and*

$$\sum_{\alpha\cdot\gamma\leq\ell} \left\|(1+\rho)^{s+\alpha\cdot\gamma} \partial^\alpha (\mathcal{S}f)\right\|_{p,\mathbb{R}^n} \leq C(\mathcal{L}, p, s) \left\|(1+\rho)^{s+\ell} f\right\|_{p,\mathbb{R}^n} \quad . \tag{6.10}$$

**Proof.** We replace $s$ by $s+\ell$ in Lemma 6.2, and deduce that the integral $\mathcal{S}f(x)$ converges absolutely for almost all $x$ in $\mathbb{R}^n$, with

$$\|(1+\rho)^s \mathcal{S}f\|_{p,\mathbb{R}^n} \leq C(\mathcal{L}, p, s) \left\|(1+\rho)^{s+\ell} f\right\|_{p,\mathbb{R}^n} \quad . \tag{6.11}$$

By Theorem 4.1 there exists a sequence $\{\varphi_k\}$ of complex $m \times 1$ vector functions in $C_0^\infty(\mathbb{R}^n)$ converging to $f$ in $W_{s+\ell}^{0,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, so that

$$\left\|(1+\rho)^{s+\ell}(\varphi_k - f)\right\|_{p,\mathbb{R}^n} \to 0 \quad . \tag{6.12}$$

Theorem 3.5 implies $\mathcal{S}\varphi_k \in C^\infty(\mathbb{R}^n)$ for each $k$, with

$$\mathcal{L}(\mathcal{S}\varphi_k) = \varphi_k \quad . \tag{6.13}$$

As (6.11) must apply also to each $\varphi_k$ and to $\varphi_k - f$, we have

$$\|(1+\rho)^s \mathcal{S}\varphi_k\|_{p,\mathbb{R}^n} \leq C(\mathcal{L}, p, s) \left\|(1+\rho)^{s+\ell}\varphi_k\right\|_{p,\mathbb{R}^n} \quad , \tag{6.14}$$

$$\|(1+\rho)^s \mathcal{S}(\varphi_k - f)\|_{p,\mathbb{R}^n} \leq C(\mathcal{L}, p, s) \left\|(1+\rho)^{s+\ell}(\varphi_k - f)\right\|_{p,\mathbb{R}^n} \quad . \tag{6.15}$$

We apply Theorem 5.2 to each function $\mathcal{S}\varphi_k$ and obtain

$$\begin{aligned}\|\mathcal{S}\varphi_k\|_{\ell,p,s;\underline{\ell}} &= \sum_{\alpha\cdot\gamma\leq\ell} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha(\mathcal{S}\varphi_k)\right\|_{p,\mathbb{R}^n} \\ &\leq C(\mathcal{L}, s, p)\left[\|(1+\rho)^s(\mathcal{S}\varphi_k)\|_{p,\mathbb{R}^n} + \left\|(1+\rho)^{s+\ell}\varphi_k\right\|_{p,\mathbb{R}^n}\right] \quad ,\end{aligned}$$

which when combined with (6.14) yields

$$\sum_{\alpha\cdot\gamma\leq\ell} \left\|(1+\rho)^{s+\alpha\cdot\gamma}\partial^\alpha(\mathcal{S}\varphi_k)\right\|_{p,\mathbb{R}^n} \leq C(\mathcal{L}, s, p)\left\|(1+\rho)^{s+\ell}\varphi_k\right\|_{p,\mathbb{R}^n} \quad . \tag{6.16}$$

But (6.16) must apply also to each difference $\varphi_k - \varphi_j$, and in view of (6.12) we conclude that the sequence $\{\mathcal{S}\varphi_k\}$ is Cauchy in the space $W_s^{\ell,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$. Hence $\{\mathcal{S}\varphi_k\}$ converges in that space to some function, which must be $\mathcal{S}f$ because of (6.15) and (6.12). In particular, $\mathcal{S}f \in W_s^{\ell,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$. Letting $k \to \infty$ in (6.16) and (6.13) gives (6.10) as well as $\mathcal{L}(\mathcal{S}f) = f$. ■

# 7 Mapping Properties

We combine our results thus far to draw conclusions about mapping properties of the partial differential operator

$$\mathcal{L} = \sum_{\alpha\cdot\gamma=\ell} A_\alpha \partial^\alpha \quad , \tag{7.1}$$

as described in the introduction.

For complex $m \times 1$ vector functions $u$ and $v$ on $\mathbb{R}^n$, we define the inner product (when it exists)

$$\langle u, v\rangle = \int_{\mathbb{R}^n} u \cdot v \; dx = \int_{\mathbb{R}^n} v^* u \; dx \quad , \tag{7.2}$$

where "*" denotes the conjugate transpose operation.

Recall that functions $u \in W_s^{0,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ and $v \in W_{-s}^{0,q}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ have the respective norms

$$\|u\|_{0,p,s;\underline{\ell}} = \|(1+\rho)^s u\|_{p,\mathbb{R}^n} \qquad , \qquad \|v\|_{0,q,-s;\underline{\ell}} = \left\|(1+\rho)^{-s} v\right\|_{q,\mathbb{R}^n} \quad .$$

If $1/p + 1/q = 1$, then (7.2) is defined for such $u$ and $v$, and according to Hölder's inequality,

$$|\langle u, v\rangle| \le \|(1+\rho)^s u\|_{p,\mathbb{R}^n} \left\|(1+\rho)^{-s} v\right\|_{q,\mathbb{R}^n} = \|u\|_{0,p,s;\underline{\ell}} \|v\|_{0,q,-s;\underline{\ell}} \quad . \tag{7.3}$$

Indeed, a standard argument confirms that, if $1 < p, q < \infty$ and $1/p + 1/q = 1$, then the spaces $W_s^{0,p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ and $W_{-s}^{0,q}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ are duals of one another.

If $u \in W_{loc}^{\underline{\ell},p}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ and $\varphi$ is a complex $m \times 1$ vector function in $C_0^\infty(\mathbb{R}^n)$, we have by the usual integration by parts that

$$\langle \mathcal{L}u, \varphi\rangle = \int_{\mathbb{R}^n} \mathcal{L}u \cdot \varphi \; dx = \int_{\mathbb{R}^n} u \cdot \mathcal{L}^* \varphi \; dx = \langle u, \mathcal{L}^* \varphi\rangle \quad , \tag{7.4}$$

where $\mathcal{L}^*$ is the *adjoint operator* to $\mathcal{L}$,

$$\mathcal{L}^* = \sum_{\alpha \cdot \gamma = \ell} (-1)^{|\alpha|} A_\alpha{}^* \partial^\alpha \quad .$$

Also, as $(\mathcal{L}^*)^* = \mathcal{L}$, we have

$$\langle \mathcal{L}^* u, \varphi\rangle = \langle u, \mathcal{L}\varphi\rangle \quad . \tag{7.5}$$

If we let $L^*(x)$ denote the symbol (1.3) for $\mathcal{L}^*$, then by a brief calculation,

$$L^*(x) = L(x)^* \quad .$$

Consequently, $L^*(x)$ is invertible whenever $L(x)$ is invertible, and semiellipticity of $\mathcal{L}$ implies the same for $\mathcal{L}^*$. Moreover, all results proved thus far for $\mathcal{L}$ are equally valid for $\mathcal{L}^*$. We let $F^*$ denote the fundamental solution for the adjoint operator $\mathcal{L}^*$,

$$F^*(x) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot z} \sigma(z) e^{-t\sigma(z)} L^*(z)^{-1} \; dz \; dt \qquad (x \neq 0) \quad ,$$

and $\mathcal{S}^*$ the corresponding convolution operator,

$$\mathcal{S}^* f(x) = F^* * f(x) = \int_{\mathbb{R}^n} F^*(x-y) f(y) \; dy \quad .$$

Obviously, Theorems 3.4, 3.5, and 6.3 apply as well to $\mathcal{L}^*$, $F^*$, and $\mathcal{S}^*$.

In accordance with (7.4) and (7.5), for complex $m \times 1$ vector functions $u$ and $f$ in $L_{loc}^1(\mathbb{R}^n)$ we say that $u$ is a *distributional solution* in $\mathbb{R}^n$ of the equation (a) $\mathcal{L}u = f$, or (b) $\mathcal{L}^* u = f$, provided that, respectively,

$$\text{(a)} \quad \langle u, \mathcal{L}^* \varphi\rangle = \langle f, \varphi\rangle \qquad , \qquad \text{(b)} \quad \langle u, \mathcal{L}\varphi\rangle = \langle f, \varphi\rangle \quad ,$$

for all complex $m \times 1$ vector functions $\varphi$ in $C_0^\infty(\mathbb{R}^n)$.

Elementary estimates confirm that, for suitable positive constants $K_1(\underline{\ell})$ and $K_2(\underline{\ell})$ and for $x \in \mathbb{R}^n$,

$$K_1(\underline{\ell})(1+|x|)^{1/\ell} \le 1+\rho(x) \le K_2(\underline{\ell})(1+|x|) \quad . \tag{7.6}$$

For $z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$ we define

$$\|z\|_x := [1+\rho(x)]^{-\ell}|z| \quad , \tag{7.7}$$

and conclude that, for another positive constant $K_3(\underline{\ell})$,

$$K_3(\underline{\ell})(1+|x|)^{-\ell}|z| \le \|z\|_x \le (1+|x|)^{\ell}|z| \quad .$$

This inequality demonstrates, according to the criterion of Hörmander ([12], § 22.1), that (7.7) defines a *temperate norm* on $\mathbb{C}^n$, parametrized by $x \in \mathbb{R}^n$. For nonzero $x \in \mathbb{R}^n$ and for $z \in \mathbb{C}^n$, use of (2.4) leads to

$$|z| = \left|L(x)^{-1}L(x)z\right| \le \left|L(x)^{-1}\right||L(x)z| \le c_4(\mathcal{L})\rho(x)^{-\ell}|L(x)z| \quad .$$

But if $\rho(x) \ge 1$ then $\rho(x)^{-\ell} \le 2^{\ell}(1+\rho(x))^{-\ell}$, and we obtain

$$|z| \le C(\mathcal{L})\|L(x)z\|_x \qquad , \qquad \text{if } \rho(x) \ge 1 \quad . \tag{7.8}$$

From (7.7) and (2.5) we deduce that

$$\begin{aligned}\|\partial^\alpha L(x)z\|_x &\le [1+\rho(x)]^{-\ell}|z| \cdot \begin{cases} c_5(\mathcal{L})\rho(x)^{\ell-\alpha\cdot\gamma} & , \text{if } \alpha\cdot\gamma \le \ell, \\ 0 & , \text{otherwise.}\end{cases} \\ &\le c_5(\mathcal{L})[1+\rho(x)]^{-\alpha\cdot\gamma}|z| \quad .\end{aligned}$$

Then with use of (7.6) and the inequality

$$\alpha\cdot\gamma = \sum_{k=1}^{n}\alpha_k\frac{\ell}{\ell_k} \ge \sum_{k=1}^{n}\alpha_k = |\alpha| \quad ,$$

we find that there is a constant $C(\mathcal{L},\alpha)$ such that

$$\|\partial^\alpha L(x)z\|_x \le C(\mathcal{L},\alpha)(1+|x|)^{-|\alpha|/\ell}|z| \quad , \tag{7.9}$$

where obviously $0 < 1/\ell \le 1$. Inequalities (7.8) and (7.9) demonstrate that $\mathcal{L}$ is a matrix *hypoelliptic operator*, as defined by Hörmander ([12], § 22.1). As a consequence (see [12]), if $f$ is of class $C^\infty$ in an open set in $\mathbb{R}^n$, then any distributional solution of $\mathcal{L}u = f$ in that open set likewise is of class $C^\infty$.

The preceding observations yield the following regularity result.

**Proposition 7.1** *Suppose $\|\gamma\| > \ell$, $1 < p < \infty$, $s \in \mathbb{R}$, $f \in W^{0,p}_{s+\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, and $u \in W^{0,p}_{s}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$. If $u$ is a distributional solution in $\mathbb{R}^n$ of $\mathcal{L}u = f$, then $u \in W^{\ell,p}_{s}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, and*

$$\|u\|_{\ell,p,s;\underline{\ell}} \le C(\mathcal{L},s,p)\left[\|(1+\rho)^s u\|_{p,\mathbb{R}^n} + \left\|(1+\rho)^{s+\ell}f\right\|_{p,\mathbb{R}^n}\right] \quad . \tag{7.10}$$

**Proof.** Let $B$ be any open ball in $\mathbb{R}^n$. As the function $f\chi_B$ is in the space $W^{0,p}_{s'+\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ for all $s' \in \mathbb{R}$, Theorem 6.3 asserts that the function $\mathcal{S}(f\chi_B)$ is in $W^{\ell,p}_{s'}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ for some $s'$, with $\mathcal{L}(\mathcal{S}(f\chi_B)) = f\chi_B$. Thus $w = u - \mathcal{S}(f\chi_B)$, a distributional solution in $B$ of $\mathcal{L}w = 0$, is in $C^\infty(B)$. As $B$ is arbitrary in $\mathbb{R}^n$, $u \in W^{\ell,p}_{loc}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$. By Theorem 5.2, $u \in W^{\ell,p}_s(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, and (7.10) holds. ■

**Lemma 7.2** *Assume* $\|\gamma\| > \ell$, $1 < p < \infty$, $1/p + 1/q = 1$, *and* $s \in \mathbb{R}$.

*a) If* $u \in W^{\ell,p}_s(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ *and* $v \in W^{\ell,q}_{-s-\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, *then*

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \qquad , \qquad \langle \mathcal{L}^* u, v\rangle = \langle u, \mathcal{L}v\rangle \quad . \tag{7.11}$$

*b) If* $f \in W^{0,q}_{-s}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$ *and*

$$-\frac{\|\gamma\|}{p} < s < \frac{\|\gamma\|}{q} - \ell \quad , \tag{7.12}$$

*then* $\mathcal{S}f$ *and* $\mathcal{S}^* f$ *are in* $W^{\ell,q}_{-s-\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, *with* $\mathcal{L}(\mathcal{S}f) = \mathcal{L}^*(\mathcal{S}^* f) = f$; *moreover, for* $u \in W^{\ell,p}_s(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$,

$$\langle \mathcal{L}u, \mathcal{S}^* f\rangle = \langle \mathcal{L}^* u, \mathcal{S}f\rangle = \langle u, f\rangle \quad . \tag{7.13}$$

**Proof.** a) By Theorem 4.1, there exists a sequence $\{\varphi_k\}$ of complex $m \times 1$ vector functions in $C_0^\infty(\mathbb{R}^n)$ converging to $v$ in the space $W^{\ell,q}_{-s-\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$; that is, with

$$\sum_{\alpha\cdot\gamma \le \ell} \left\| (1+\rho)^{-s-\ell+\alpha\cdot\gamma} \partial^\alpha (v - \varphi_k) \right\|_{q,\mathbb{R}^n} \longrightarrow 0 \quad .$$

Since $u \in W^{\ell,p}_{loc}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, for each $\varphi_k$ we have

$$\langle \mathcal{L}u, \varphi_k\rangle = \langle u, \mathcal{L}^* \varphi_k\rangle \quad .$$

Since also $\|(1+\rho)^s u\|_{p,\mathbb{R}^n} < \infty$ and $\left\|(1+\rho)^{s+\ell} \mathcal{L}u\right\|_{p,\mathbb{R}^n} < \infty$, in view of (7.3) we may let $k \to \infty$ in this equation to obtain the left equation of (7.11). The right equation of (7.11) follows similarly, or by replacing $\mathcal{L}$ with $\mathcal{L}^*$.

b) We apply Theorem 6.3 but with $p$ replaced by $q$ and $s$ replaced by $-s-\ell$. The hypothesis (6.9) is replaced by

$$-\|\gamma\|/q < -s - \ell < \|\gamma\| - \ell - \|\gamma\|/q \quad ,$$

which follows from (7.12). The theorem concludes that $\mathcal{S}f$ and $\mathcal{S}^* f$ are in $W^{\ell,q}_{-s-\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, with $\mathcal{L}(\mathcal{S}f) = \mathcal{L}^*(\mathcal{S}^* f) = f$. Application of (7.11) to $v = \mathcal{S}^* f$ and $v = \mathcal{S}f$ yields (7.13). ■

**Theorem 7.3** *Assume $\|\gamma\| > \ell$, $1 < p < \infty$, $1/p+1/q = 1$, $s \in \mathbb{R}$, and consider the mapping*

$$\mathcal{L} : W_s^{\ell,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \longrightarrow W_{s+\ell}^{0,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \quad . \tag{7.14}$$

*a) If $-\|\gamma\|/p < s$, then the mapping is one-to-one.*
*b) If $-\|\gamma\|/p < s < \|\gamma\|/q - \ell$, the mapping is onto,*
*c) If $s < -\|\gamma\|/p$, the mapping is not one-to-one.*
*d) If $s \geq \|\gamma\|/q - \ell$ the mapping is not onto.*
*e) If $s = -\|\gamma\|/p$, the mapping is not bounded below.*
*Consequently, (7.14) is an isomorphism if and only if*

$$-\|\gamma\|/p < s < \|\gamma\|/q - \ell \quad . \tag{7.15}$$

**Proof.** a) Under the given assumptions, assume $u \in W_s^{\ell,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ and $\mathcal{L}u = 0$. We may choose $s$ smaller, if necessary, so that (7.12) holds. Then by Lemma 7.2(b), for all $f$ in $W_{-s}^{0,q}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ we have

$$\langle u, f\rangle = \langle \mathcal{L}u, \mathcal{S}^* f\rangle = \langle 0, \mathcal{S}^* f\rangle = 0 \quad .$$

As $u \in W_s^{0,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ and $W_{-s}^{0,q}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ is the dual space, we infer that $u = 0$. Thus (7.14) is one-to-one.

b) Under the given assumptions, let $f \in W_{s+\ell}^{0,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$. By Theorem 6.3, $\mathcal{S}f \in W_s^{\ell,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ and $\mathcal{L}\left(\mathcal{S}f\right) = f$. Thus $\mathcal{L}$ is onto.

c) Since $\mathcal{L}$ has no zero order term, any constant vector $u$ solves $\mathcal{L}u = 0$. But for such $u$, (4.2) gives

$$\|u\|_{\ell,p,s;\underline{\ell}} = \left\|(1+\rho)^s u\right\|_{p,\mathbb{R}^n} = |u| \left(\int_{\mathbb{R}^n} (1+\rho)^{sp} \, dx\right)^{1/p} \quad ,$$

which according to Lemma 3.1 is finite whenever $sp < -\|\gamma\|$. Thus, if $s < -\|\gamma\|/p$, then $u \in W_s^{\ell,p}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ and (7.14) is not one-to-one.

d) We consider the case $s > \|\gamma\|/q - \ell$, but delay the case $s = \|\gamma\|/q - \ell$ until after the proof of (e). Let $v$ be any nonzero $m \times 1$ constant function, and let $f$ be the function

$$f(x) = \left[1 + \rho(x)\right]^{-(s+\ell+\|\gamma\|)} v \quad .$$

By Lemma 3.1,

$$\|v\|_{\ell,q,-s-\ell;\underline{\ell}} = |v| \left(\int_{\mathbb{R}^n} (1+\rho)^{-(s+\ell)q} \, dx\right)^{1/q} < \infty \quad ,$$

since $-(s+\ell)q < -\|\gamma\|$; also,

$$\begin{aligned} \|f\|_{0,p,s+\ell;\underline{\ell}} &= \left(\int_{\mathbb{R}^n} (1+\rho)^{(s+\ell)p} |f|^p \, dx\right)^{1/p} \\ &= |v| \left(\int_{\mathbb{R}^n} (1+\rho)^{-\|\gamma\|p} \, dx\right)^{1/p} < \infty \quad , \end{aligned}$$

as $-\|\gamma\|\,p < -\|\gamma\|$. Thus $v \in W^{\ell,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ and $f \in W^{0,p}_{s+\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$. If $f = \mathcal{L}u$ for some $u$ in $W^{\ell,p}_{s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$, then Lemma 7.2(a) requires that

$$\begin{aligned} 0 &= \langle u, 0\rangle = \langle u, \mathcal{L}^* v\rangle = \langle \mathcal{L}u, v\rangle = \langle f, v\rangle \\ &= \int_{\mathbb{R}^n} f \cdot v \; dx = |v|^2 \int_{\mathbb{R}^n} \left[1+\rho\left(x\right)\right]^{-(s+\ell+\|\gamma\|)} \; dx > 0 \quad , \end{aligned}$$

a contradiction. Thus (7.14) is not onto.

e) Let $s = -\|\gamma\|/p$. Given $R \geq 1$, let $\psi$ be the function described in the proof of Theorem 4.1, satisfying $\psi \in C_0^\infty\left(\mathbb{R}^n\right)$, $0 \leq \psi \leq 1$, $\psi\left(x\right) = 1$ for $\rho\left(x\right) \leq R$, $\psi\left(x\right) = 0$ for $\rho\left(x\right) \geq 2R$, and

$$\left|\partial^\alpha \psi\left(x\right)\right| \leq C\left(\ell, \alpha\right) R^{-\alpha\cdot\gamma} \quad .$$

Let $v$ be any nonzero constant $m \times 1$ vector, and let $u$ be the function $u\left(x\right) = \psi\left(x\right) v$. Then

$$\left|\mathcal{L}u\left(x\right)\right| = \left|\sum_{\alpha\cdot\gamma=\ell} A_\alpha \partial^\alpha \psi\left(x\right) v\right| \leq \begin{cases} C\left(\mathcal{L}\right)|v|\,R^{-\ell} & \text{, if } R \leq \rho\left(x\right) \leq 2R\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

As $1 \leq R \leq \rho \leq 2R$ implies $\rho \leq 1+\rho \leq 2\rho$, we have

$$\left\|\left(1+\rho\right)^{s+\ell} \mathcal{L}u\right\|_{p,\mathbb{R}^n} \leq C\left(\mathcal{L}\right)|v|\,R^{-\ell}\left(\int_{R\leq\rho(x)\leq 2R} \rho\left(x\right)^{(s+\ell)p} \; dx\right)^{1/p} \quad .$$

In the last integral we set $x = R^\gamma y$, $\rho\left(x\right) = R\rho\left(y\right)$, $dx = R^{\|\gamma\|}dy$ and obtain

$$\begin{aligned} \int_{R\leq\rho(x)\leq 2R} \rho\left(x\right)^{(s+\ell)p} \; dx &= R^{(s+\ell)p+\|\gamma\|} \int_{1\leq\rho(y)\leq 2} \rho\left(y\right)^{(s+\ell)p} \; dy \\ &= C\left(\mathcal{L}, p\right) R^{(s+\ell)p+\|\gamma\|} \quad , \end{aligned}$$

and thereby, upon setting $s = -\|\gamma\|/p$,

$$\left\|\left(1+\rho\right)^{s+\ell} \mathcal{L}u\right\|_{p,\mathbb{R}^n} \leq C\left(\mathcal{L}, p\right)|v| \quad . \tag{7.16}$$

On the other hand, Lemma 3.1 ensures that, as $R \to \infty$,

$$\begin{aligned} \left\|\left(1+\rho\right)^s u\right\|_{p,\mathbb{R}^n} &\geq \left(\int_{\rho(x)\leq R} \left[1+\rho\left(x\right)\right]^{sp} \left|u\left(x\right)\right|^p \; dx\right)^{1/p} \\ &= |v|\left(\int_{\rho(x)\leq R} \left[1+\rho\left(x\right)\right]^{-\|\gamma\|} \; dx\right)^{1/p} \longrightarrow \infty \quad . \end{aligned}$$

Combining this result with (7.16), we conclude that, as $R \to \infty$,

$$\frac{\|\mathcal{L}u\|_{0,p,s+\ell;\underline{\ell}}}{\|u\|_{\ell,p,s;\underline{\ell}}} \leq \frac{C\left(\mathcal{L}, p\right)|v|}{\left\|\left(1+\rho\right)^s u\right\|_{p,\mathbb{R}^n}} \longrightarrow 0 \quad .$$

By choosing $R$ large enough we can make the left side of this expression as small as desired; thus the mapping (7.14) is not bounded below in the case $s = -\|\gamma\|/p$.

We are left with only the case $s = \|\gamma\|/q - \ell$. We assume (7.14) is onto, and we will produce a contradiction. The condition $\|\gamma\| > \ell$ implies $s > -\|\gamma\|/p$, and the result of (a) confirms the mapping is one-to-one. Consequently, as a bounded, onto, one-to-one.mapping from one Banach space to another, $\mathcal{L}$ has a bounded inverse mapping $\mathcal{L}^{-1}$. In particular,

$$\mathcal{L}^{-1} : W^{0,p}_{s+\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \longrightarrow W^{\ell,p}_{s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$$

and there is a positive constant $M$ such that

$$\left\|\mathcal{L}^{-1} v\right\|_{\ell,p,s;\underline{\ell}} \leq M \|v\|_{0,p,s+\ell;\underline{\ell}} \quad .$$

Given any function $f$ in $W^{0,q}_{-s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$, we may define a mapping $T$ from $W^{0,p}_{s+\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ into $\mathbb{C}$, according to

$$T(v) = \left\langle \mathcal{L}^{-1} v, f \right\rangle \quad .$$

Obviously $T$ is linear, and it is also bounded, as verified by

$$\begin{aligned} |T(v)| &= \left|\left\langle \mathcal{L}^{-1} v, f \right\rangle\right| \leq \left\|\mathcal{L}^{-1} v\right\|_{0,p,s;\underline{\ell}} \|f\|_{0,q,-s;\underline{\ell}} \\ &\leq \left\|\mathcal{L}^{-1} v\right\|_{\ell,p,s;\underline{\ell}} \|f\|_{0,q,-s;\underline{\ell}} \leq M \|f\|_{0,q,-s;\underline{\ell}} \|v\|_{0,p,s+\ell;\underline{\ell}} \quad . \end{aligned}$$

Since $W^{0,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ is the dual of $W^{0,p}_{s+\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$, there exists a function $u$ in $W^{0,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ such that

$$T(v) = \left\langle \mathcal{L}^{-1} v, f \right\rangle = \langle v, u \rangle \qquad , \qquad \forall v \in W^{0,p}_{s+\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \quad .$$

Giving any complex $m \times 1$ vector function $\varphi$ in $C^\infty_0\left(\mathbb{R}^n\right)$ we may choose $v = \mathcal{L}\varphi$ to obtain

$$\langle \varphi, f \rangle = \langle \mathcal{L}\varphi, u \rangle \qquad , \qquad \langle f, \varphi \rangle = \langle u, \mathcal{L}\varphi \rangle \quad .$$

This relation implies that $u$ is a distributional solution in $\mathbb{R}^n$ of the equation $\mathcal{L}^* u = f$. By Proposition 7.1 applied to $\mathcal{L}^*$, and with $p$ replaced by $q$ and $s$ by $-s-\ell$, we have $u \in W^{\ell,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$. As $f$ is arbitrary in $W^{0,q}_{-s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$, this argument shows that the mapping

$$\mathcal{L}^* : W^{\ell,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right) \longrightarrow W^{0,q}_{-s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$$

is onto. But the result of (e), with $\mathcal{L}$ replaced by $\mathcal{L}^*$, $p$ by $q$, and $s$ by $-s-\ell$, applies to this mapping, as $-s-\ell = -\|\gamma\|/q$. Since $\mathcal{L}^*$ is onto but not bounded below, it cannot be one-to-one. Hence there is a function $w$ in $W^{\ell,q}_{-s-\ell}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$ such that $w \neq 0$ and $\mathcal{L}^* w = 0$. By Lemma 7.2(a) we then have, for all $u$ in $W^{\ell,p}_{s}\left(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}\right)$,

$$\langle \mathcal{L}u, w \rangle = \langle u, \mathcal{L}^* w \rangle = \langle u, 0 \rangle = 0 \quad .$$

But we assume (7.14) is onto, so we have $\langle f, w\rangle = 0$ for all $f$ in $W^{0,p}_{s+\ell}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell})$, a contradiction if $w \neq 0$. ■

As a special case of Theorem 7.3, we take $s = -\ell$ and find that the mapping

$$\mathcal{L} : W^{\ell,p}_{-\ell}(\mathbb{R}^n \mathbb{C}^m, \underline{\ell}) \longrightarrow W^{0,p}_{0}(\mathbb{R}^n, \mathbb{C}^m, \underline{\ell}) \tag{7.17}$$

is an isomorphism provided that $\ell < \|\gamma\| / p$. This result has already been obtained by Demidenko [3, 4], who wrote (7.17) with the notation

$$\mathcal{L} : W^{\underline{\ell}}_{p,1}(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n) \quad .$$

# 8 Examples

We give a few examples to which results of the paper apply.

**Example 8.1** *Consider a parabolic opertor in* $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$,

$$\mathcal{L}u = \sum_{|\alpha|=\ell} A_\alpha \partial_x^\alpha u - I \partial_t u \quad , \tag{8.1}$$

*where each* $A_\alpha$ *is a complex constant* $m \times m$ *matrix,* $I$ *is the* $m \times m$ *identity, and* $u = u(x_1, \ldots, x_n, t)$*. The usual parabolicity condition (see [8]) requires that each eigenvalue* $\lambda(x)$ *of the matrix*

$$P(x) = \sum_{|\alpha|=\ell} A_\alpha (ix)^\alpha$$

*satisfy an inequality*

$$\operatorname{Re}\ \lambda(x) \leq -\delta |x|^\ell \qquad (\delta > 0) \quad . \tag{8.2}$$

*For (8.1), formula (1.3) gives*

$$L(x,t) = \sum_{|\alpha|=\ell} A_\alpha (ix)^\alpha - itI \quad .$$

*Thus* $L(x,t)$ *is invertible if* $(x,t) \neq 0$*, as (8.2) shows that* $P(x)$ *has no purely imaginary eigenvalue when* $x \neq 0$*. Also for (8.1), whose order is* $\ell$*, we determine that*

$$\underline{\ell} = (\ell, \ldots, \ell, 1) \qquad , \qquad \gamma = (1, \ldots, 1, \ell) \qquad , \qquad \|\gamma\| = n + \ell > \ell \quad ,$$

$$\rho(x,t) = \left( t^2 + \sum_{k=1}^{n} {x_k}^{2\ell} \right)^{1/2\ell} \quad .$$

*Thus Theorem 7.3 applies, and we conclude that the mapping (7.14) is an isomorphism if and only if*

$$-\frac{n+\ell}{p} < s < n - \frac{n+\ell}{p} \quad .$$

*For the special case of the heat equation,*

$$\mathcal{L}u = \Delta u - \partial u/\partial t \quad ,$$

*this result was proved in [10].*

**Example 8.2** *Consider in $\mathbb{R}^n$ the operator*

$$\mathcal{L}u = \sum_{|\alpha|=\ell} A_\alpha \partial_x^\alpha u \quad ,$$

*where again each $A_\alpha$ is a complex $m \times m$ matrix. The symbol is*

$$L(x) = \sum_{|\alpha|=\ell} A_\alpha (ix)^\alpha \quad ,$$

*and the semiellipticity requirement that $L(x)$ be invertible if $x \neq 0$ reduces to the usual requirement for ellipticity of the operator. We have*

$$\underline{\ell} = (\ell, \ell, \ldots, \ell) \qquad , \qquad \gamma = (1, 1, \ldots, 1) \qquad , \qquad \|\gamma\| = n \quad ,$$

$$\rho(x) = \left( \sum_{k=1}^{n} {x_k}^{2\ell} \right)^{1/2\ell} \qquad .$$

*Obviously $\rho(x)$ is equivalent to the simpler weight function $|x|$. Theorem 7.3 applies only if $\|\gamma\| = n > \ell$, in which case the mapping (7.14) is an isomorphism if and only if*

$$-\frac{n}{p} < s < n - \ell - \frac{n}{p} \quad .$$

**Example 8.3** *Let $k$ and $r$ be positive integers, and let $\mathcal{L}$ be the operator*

$$\mathcal{L}u = \sum_{j=0}^{k} \sum_{|\beta|=jr} A_{\beta,k-j} \partial_x^\beta \partial_t^{k-j} u \quad , \tag{8.3}$$

*where again $u = u(x_1, \ldots, x_n, t)$, and each $A_\beta$ is a complex constant $m \times m$ matrix. The semiellipticity condition is that the matrix*

$$\sum_{j=0}^{k} \sum_{|\beta|=jr} A_{\beta,k-j} (ix)^\beta (it)^{k-j} \tag{8.4}$$

*be invertible when $(x,t) \neq 0$. The order of this operator is $\ell = kr$, while*

$$\underline{\ell} = (kr, \ldots, kr, k) \qquad , \qquad \gamma = (1, \ldots, 1, r) \qquad , \qquad \|\gamma\| = n + r \quad ,$$

$$\rho(x,t) = \left(t^{2k} + \sum_{j=1}^{n} {x_k}^{2kr}\right)^{1/2kr} \quad .$$

*Our condition* $\|\gamma\| > \ell$ *reduces to* $n > (k-1)\,r$, *and the isomorphism condition (7.15) becomes*

$$-\frac{n+r}{p} < s < n - (k-1)\,r - \frac{n+r}{p} \quad .$$

*In the scalar case* $m = 1$, *and with* $A_{0,k} = i^{-k}$, *(8.4) becomes*

$$f(x,t) := t^k + \sum_{j=1}^{k} \sum_{|\beta|=jr} A_{\beta,k-j}\,(ix)^{\beta}\,(it)^{k-j} \quad .$$

*The operator* $\mathcal{L}$ *in this case is said to be "r-parabolic" [13, 18] under the additional assumption that* $\operatorname{Im}\ t \geq \delta > 0$ *for each root* $t$ *of* $f(x,t) = 0$, *as* $x$ *ranges over* $\mathbb{R}^n$ *with* $|x| = 1$.

# References


[1] M. Cantor, Spaces of functions with asymptotic conditions on $R^n$, Indiana Univ. Math. J. 24 (1975), 897-902.

[2] G. V. Demidenko, On weighted Sobolev spaces and integral operators determined by quasi-elliptic equations, Russian Acad. Sci. Dokl. Math. 49 (1994), 113-118.

[3] G. V. Demidenko, On quasielliptic operators in $R^n$, Siberian Math. J. 39 (1998), 884-893.

[4] G. V. Demidenko, Isomorphic properties of quasi-elliptic operators, Doklady Mathematics 59 (1999), 102-106.

[5] G. V. Demidenko, Isomorphic properties of one class of differential operators and their applications, Siberian Math. J. 42 (2001), 865-883.

[6] G. V. Demidenko, On properties of a class of matrix differential operators in $R^n$, Selçuk J. Appl. Math. 3 (2002), 23-36.

[7] G. V. Demidenko, Isomorphic properties of one class of matrix differential operators, Doklady Mathematics 68 (2003), 4-7.

[8] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[9] D. T. Haimo and C. Markett, A representation theory for solutions of a higher order heat equation, I, J. Math. Anal. Appl. 168 (1992), 89-107.

[10] G. N.Hile and C. P. Mawata, The behaviour of the heat operator on weighted Sobolev spaces, Trans. Amer. Math. Soc. 350, no. 4 (1998), 1407-1428.

[11] G. N. Hile, C. P. Mawata, and C. Zhou, Apriori bounds for semielliptic operators, J. Differential Equations, vol. 176 (2001), 29-64.

[12] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Springer-Verlag, 1985.

[13] M. Itano, Some remarks on the Cauchy problem for p-parabolic equations, Hiroshima Math. J. 4 (1974), 211-228.

[14] R. B. Lockhart, Fredholm properties of a class of elliptic operators on non-compact manifolds, Duke Math. J. 48 (1981), 289-312.

[15] R. B. Lockhart and R. C. McOwen, On elliptic systems in $R^n$, Acta. Math. 150 (1983), 125-135.

[16] R. C. McOwen, The behavior of the Laplacian on weighted Sobolev Spaces, Comm. Pure Appl. Math. 32 (1979), 783-795.

[17] R. C. McOwen, On elliptic operators in $R^n$, Comm. Partial Differential Equations 5 (1980), 913-933.

[18] S. Mizohata, Le problème de Cauchy pour les équations paraboliques, J. Math. Soc. Japan 8 (1956), 269-299.

[19] L. Nirenberg and H. F. Walker, The null spaces of elliptic partial differential operators in $R^n$, J. Math. Anal. Appl. 42 (1973), 271-301.

[20] S. V. Uspenskiĭ, On the representation of functions defined by a class of hypoelliptic operators, Trudy Mat. Inst. Steklov Akad. Nauk 117 (1972), 292-299; English transl. in Proc. Steklov Inst. Math. 117 (1972), 343-352.

[21] H. F. Walker, On the null-spaces of first-order elliptic partial differential operators in $R^n$, Proc. Amer. Math. Soc. 30 (1971), 278-286.

[22] H. F. Walker, On the null-spaces of elliptic partial differential operators in $R^n$, Trans. Amer. Math. Soc. 173 (1972), 263-275.